Space charge algorithm for the Multi Ensemble Model

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Abstract. Derived from the Vlasov equation the Ensemble Model [1] has been elaborated for fast and efficient beam dynamics simulations. The Model represents a particle beam as a set of sub-beams or Ensembles, described by coordinates of the centroid and 6D phase space correlations. Whereas a space charge routine for the Single Ensemble Model (SEM) has been developed and tested [2], implementation of the space charge algorithm for the Multi Ensemble Model (MEM) needs more efforts. A space charge model based on the Multi-Centered Gaussian Expansion (MCGE) [3] implies a smooth particle density distribution within an Ensemble but it requires rather large computational efforts. This paper presents another space charge algorithm, based on the analytical solution for the electrical field of an ellipsoidal 3D charge distribution [4]. Using this algorithm one can calculate the space charge force and its gradient inside and outside driving Ensemble. Features of the implementation and simplifying approximations are discussed in this paper.

1. Introduction

The Ensemble Model [1] describes a particle beam by set of sub-beams or Ensembles, each of them is characterized not only by average position, but also by correlations (or moments of the Ensemble distribution function) in 6D phase space. Considering the moments up to second order one can describe each Ensemble by 27 parameters (6 moments of the first and 21 moments of the second order). This implies linear external and internal forces inside an Ensemble and therefore, Ensemble emittance invariance [1]. To simulate the beam emittance the Multi Ensemble Model (MEM) [2] can be used; nonlinear Lorenz force effects can be modeled by a set of Ensembles distributed in the phase space. Whereas external force implementation in MEM is straightforward, the space charge force algorithm needs more efforts.

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2. Main Equations of the Ensemble Model

A beam distribution function could be represented as a superposition of distribution functions of Ensembles:

$$\Psi(\vec{r},\vec{p}) = \sum_n w_n \psi_n(\vec{r},\vec{p}), \tag{1}$$

where w_n is related to the Ensemble charge weight function ($\sum_n w_n = 1$). Each Ensemble is described by 6 first order moments of the distribution function $\psi_n(\vec{r},\vec{p})$:

$$\begin{aligned} \langle \xi \rangle &= \int \xi \, \psi_n(\vec{r},\vec{p}) d\vec{r} d\vec{p} \\ \langle p_\nu \rangle &= \int p_\nu \, \psi_n(\vec{r},\vec{p}) d\vec{r} d\vec{p}, \end{aligned} \tag{2}$$

and by 21 moments of the second order:

$$\begin{aligned} M_{\xi\nu} &= \langle \Delta\xi \cdot \Delta\nu \rangle = \int \Delta\xi \cdot \Delta\nu \cdot \psi_n(\vec{r},\vec{p}) d\vec{r} d\vec{p} \\ M_{\xi p_\nu} &= \langle \Delta\xi \cdot \Delta p_\nu \rangle = \int \Delta\xi \cdot \Delta p_\nu \cdot \psi_n(\vec{r},\vec{p}) d\vec{r} d\vec{p} \\ M_{p_\xi p_\nu} &= \langle \Delta p_\xi \cdot \Delta p_\nu \rangle = \int \Delta p_\xi \cdot \Delta p_\nu \cdot \psi_n(\vec{r},\vec{p}) d\vec{r} d\vec{p}, \end{aligned} \tag{3}$$

where $\xi,\nu = \{x,y,z\}$ and $\Delta\xi = \xi - \langle\xi\rangle, \Delta\nu = \nu - \langle\nu\rangle$.

Assuming Ensemble energy spread small, considering all moments of the distribution function till the second order, one can obtain main equation for any Ensemble parameter $\mu = \{\xi, p_\nu, \Delta\xi\cdot\Delta\nu, \Delta\xi\cdot\Delta p_\nu, \Delta p_\xi \cdot \Delta p_\nu\}$ [1,2]:

$$\frac{\partial\langle\mu\rangle}{\partial\tau} = \left\langle \frac{\partial\mu}{\partial\vec{r}} \frac{\vec{p}}{\gamma} \right\rangle + \left\langle \frac{\partial\mu}{\partial\vec{p}} \frac{\vec{F}}{mc^2} \right\rangle, \tag{4}$$

where $\gamma = \sqrt{1+\vec{p}\cdot\vec{p}}$ is normalized energy, $\vec{p} = \vec{P}/(mc)$ is normalized momentum, $\tau = ct$ and $\vec{F}$ is applied Lorentz force.

Using Lorentz force expansion till linear terms

$$\frac{\vec{F}}{mc^2} = \vec{\mathbf{F}}(\langle\vec{r}\rangle,\langle\vec{p}\rangle) + \hat{\mathbf{F}}^X \cdot \Delta\vec{r} + \hat{\mathbf{F}}^P \cdot \Delta\vec{p}, \tag{5}$$

where matrices

$$\hat{\mathbf{F}}^X_{ij} = \frac{1}{mc^2} \left.\frac{\partial \vec{F}_i}{\partial r_j}\right|_{\langle\vec{r}\rangle,\langle\vec{p}\rangle}, \quad \hat{\mathbf{F}}^P_{ij} = \frac{1}{mc^2} \left.\frac{\partial \vec{F}_i}{\partial p_j}\right|_{\langle\vec{r}\rangle,\langle\vec{p}\rangle}, \tag{6}$$

one can obtain 6 time equations for the first order moments $\langle\vec{r}\rangle, \langle\vec{p}\rangle$:

$$\begin{aligned} \frac{d\langle \vec{p} \rangle}{d\tau} &= \vec{\mathbf{F}}\left(\langle \vec{r} \rangle, \langle \vec{p} \rangle\right) \\ \frac{d\langle \vec{r} \rangle}{d\tau} &= \hat{\mathbf{W}} \cdot \langle \vec{p} \rangle, \end{aligned} \tag{7}$$

and 21 equations for the second order moments:

$$\begin{aligned} \frac{d\hat{\mathbf{M}}_{pp}}{d\tau} &= \hat{\mathbf{F}}^{X} \cdot \hat{\mathbf{M}}_{xp} + \hat{\mathbf{F}}^{P} \cdot \hat{\mathbf{M}}_{pp} + \left(\hat{\mathbf{F}}^{X} \cdot \hat{\mathbf{M}}_{xp} + \hat{\mathbf{F}}^{P} \cdot \hat{\mathbf{M}}_{pp}\right)^{T}, \\ \frac{d\hat{\mathbf{M}}_{xp}}{d\tau} &= \hat{\mathbf{V}} \cdot \hat{\mathbf{M}}_{pp} + \hat{\mathbf{M}}_{xx} \cdot \left(\hat{\mathbf{F}}^{X}\right)^{T} + \hat{\mathbf{M}}_{xp} \cdot \left(\hat{\mathbf{F}}^{P}\right)^{T}, \\ \frac{d\hat{\mathbf{M}}_{xx}}{d\tau} &= \hat{\mathbf{M}}_{xp} \cdot \hat{\mathbf{V}} + \hat{\mathbf{V}} \cdot \hat{\mathbf{M}}_{xp}^{T}. \end{aligned} \tag{8}$$

Elements of the auxiliary matrices $\hat{\mathbf{W}}, \hat{\mathbf{V}}$, used in (7) and (8) are

$$\begin{aligned} \hat{\mathbf{W}}_{ij} &= \frac{1}{\gamma_m}\left(\delta_{ij} - \frac{\hat{\mathbf{M}}_{p_i p_j}}{\gamma_m^2}\right) \\ \hat{\mathbf{V}}_{ij} &= \frac{1}{\gamma_m}\left(\delta_{ij} - \frac{\langle p_i \rangle \langle p_j \rangle}{\gamma_m^2}\right) \end{aligned}, \tag{9}$$

where $\gamma_m^2 = 1 + \sum_{n=x,y,z}\left(\langle p_n \rangle^2 + \hat{\mathbf{M}}_{p_n p_n}\right)$ is squared normalized Ensemble energy.

3. Space Charge Implementation

Since the Ensemble Model implies internal motion even in the case of the Single Ensemble Model (SEM) collective effects in beam dynamics can be simulated, despite the beam emittance remains constant in the SEM, there is a good agreement in beam size and beam divergence simulation [2].

The space charge implementation makes an Ensemble charge distribution function an important issue. The rigorous problem reduces to the determining the stationary 3-D charge distribution (which does not explicitly depend on time), which corresponds to the linear applied forces. The distribution in which the forces are linear and the phase space areas remain constant is known as microcanonical distribution [5]. A homogeneous (x, y) ellipsoidal beam distribution, known as K-V distribution leads to a perfect linear space charge force within the beam radius. The space charge model for the Single Ensemble Model (SEM) is based on the homogeneously charged ellipsoid. Calculation of the Lorenz force gradient at the Ensemble center reduced to obtaining the resulting force at a small offset from the homogeneously charged ellipsoid, one of the aproaches is integration over thin shell of uncompensated charges [1].

In the case of several Ensembles (MEM) it is necessary to calculate not only the space charge gradient at the center of the driving Ensemble, but also Lorentz force and its gradient at positions of others Ensembles. The most probable macroparticles configuration is a set of overlapping Ensembles. One of the algorithms, based on distribution function expansion is Multi-Centered Gaussian Expansion (MCGE) is discussed in [3]. This approach is based on the expansion of the Ensemble charge density in distributed basis functions with known solutions of the field equation. The main advantage of this algorithm is a smoothness of distribution function, but nessesety of solution of linear equation system on each integration step for each Ensemble makes this approach comparatively slow.

A model of homogeneously charged 3D ellipsoid being very useful for the calculation of the space charge force gradient in the SEM can be extended to MEM. The uniform ellipsoidal is not a solution of the Poisson-Vlasov system because the corresponding stationary distribution in the phase space is singular; nevertheless, it allows one to keep Hamiltonian character of the 3D model similar to the dynamics of the full particle system (Liouville problem) [4].

4. Space Charge Field of an Ellipsoidal Ensemble Distribution

Distribution function of a 3-axis homogeneous ellipsoidal Ensemble is given by

$$\psi(x,y,z)=\begin{cases}\dfrac{3\gamma}{4\pi s_x s_y s_z}, & \text{if} \quad \dfrac{x^2}{s_x^2}+\dfrac{y^2}{s_y^2}+\dfrac{\gamma^2 z^2}{s_z^2}\le 1\\ 0, \quad \text{if} & \dfrac{x^2}{s_x^2}+\dfrac{y^2}{s_y^2}+\dfrac{\gamma^2 z^2}{s_z^2}>1\end{cases}, \tag{10}$$

where ellipsoid semi-axes and rms sizes are defined by matrix elements (3): $s_x^2=5\sigma_x^2=5M_{xx}$, $s_y^2=5\sigma_y^2=5M_{yy}$, $s_z^2=5\gamma\sigma_z^2=5\gamma^2 M_{zz}$. From Newton's potential theory [6] the electric field $\vec{E}$ in the Ensemble's rest frame is given:

$$E_i=\frac{qr_i}{s_x s_y s_z}\tilde{G}_i\left(a_x,a_y,a_z\right), \tag{11}$$

where q denotes the Ensemble charge, $a_i=s_i^2+\lambda$ is a square of the equivalent confocal ellipsoid semi-axis, parameter $\lambda=0$ for internal point of the ellipsoid, otherwise, for an external point λ can be determined as the positive root of the equation:

$$\frac{x^2}{s_x^2+\lambda}+\frac{y^2}{s_y^2+\lambda}+\frac{\gamma^2 z^2}{s_z^2+\lambda}=1. \qquad (12)$$

This equation determines one and only one ellipsoid passes through any point (x,y,z) outside the ellipsoid. A geometrical form factor $\tilde{G}_i$ ($\{i,j,k\}$ defines any permutation of the indices $\{x,y,z\}$):

$$\tilde{G}_i=\frac{s_x s_y s_z}{\sqrt{a_x a_y a_z}}G\left(\frac{a_i}{a_j},\frac{a_i}{a_k}\right),\ \text{where}\ \ G(p,q)=3\cdot\int_0^1\frac{v^2dv}{\sqrt{(p+(1-p)v^2)(q+(1-q)v^2)}} \qquad (13)$$

is constant for internal point of the ellipsoid, so as is well known, the electric field is linear inside the ellipsoid. For external point $a_i=f_i(x,y,z)$ and $\tilde{G}_i$ determines field decay with an offset from the Ensemble center. Equation (12) can be interpreted in the following way: at any external point the electric field generated by an ellipsoidal uniform charge distribution is equivalent to the electric field generated by a confocal uniformly charged ellipsoid passing through the point (x,y,z).

The geometrical form factor as a consequence of the Gaussian theorem for the electric field satisfies the equality:

$$G\left(\frac{a_x}{a_y},\frac{a_x}{a_z}\right)+G\left(\frac{a_y}{a_x},\frac{a_y}{a_z}\right)+G\left(\frac{a_z}{a_x},\frac{a_z}{a_y}\right)=3 \qquad (14)$$

Assuming that the driving Ensemble has energy $E=mc^2\gamma=mc^2\big/\sqrt{1-\beta^2}$, and center coordinates $\langle\vec{r}\rangle=\{x^d,y^d,z^d\}$, the Lorenz force acting from the driving Ensemble on the test particle $\{x^t,y^t,z^t,\beta_x^t=v_x^t/c,\beta_y^t=v_y^t/c,\beta_z^t=v_z^t/c\}$ is given by:

$$\begin{aligned}
F_x^{d\to t}&=e\gamma(1-\beta\beta_z^t)\cdot E_x^d\left(\vec{\Delta}^{dt}\right)\approx eE_x^d\left(\vec{\Delta}^{dt}\right)/\gamma=\frac{eq\cdot\left(x^t-x^d\right)}{5^{3/2}\gamma^2\sigma_x\sigma_y\sigma_z}\tilde{G}_x,\\
F_y^{d\to t}&=e\gamma(1-\beta\beta_z^t)\cdot E_y^d\left(\vec{\Delta}^{dt}\right)\approx eE_y^d\left(\vec{\Delta}^{dt}\right)/\gamma=\frac{eq\cdot\left(y^t-y^d\right)}{5^{3/2}\gamma^2\sigma_x\sigma_y\sigma_z}\tilde{G}_y, \qquad (15)\\
F_z^{d\to t}&=eE_z^d\left(\vec{\Delta}^{dt}\right)+e\beta\gamma\left(\beta_x^t E_x^d\left(\vec{\Delta}^{dt}\right)+\beta_y^t E_y^d\left(\vec{\Delta}^{dt}\right)\right)\approx eE_z^d\left(\vec{\Delta}^{dt}\right)=\frac{eq\cdot\left(z^t-z^d\right)}{5^{3/2}\sigma_x\sigma_y\sigma_z}\tilde{G}_z,
\end{aligned}$$

where vector $\vec{\Delta}^{dt}=\{x^d-x^t,y^d-y^t,\left(z^d-z^t\right)\cdot\gamma\}$ takes into account Lorentz transformation for the coordinates. The matrix $\hat{\mathbf{F}}^X$ is given by

$$\hat{\mathbf{F}}^X=\frac{eq}{mc^2}\frac{1}{\gamma^2\sigma_x\sigma_y\sigma_z}\left\{\begin{pmatrix}\tilde{G}_x & 0 & 0\\ 0 & \tilde{G}_y & 0\\ 0 & 0 & \gamma^2\tilde{G}_z\end{pmatrix}+\begin{pmatrix}\left(x^t-x^d\right)\cdot\tilde{G}_x^1\\ \left(y^t-y^d\right)\cdot\tilde{G}_y^1\\ \left(z^t-z^d\right)\gamma^2\tilde{G}_z^1\end{pmatrix}\times\begin{pmatrix}\frac{\partial\lambda}{\partial x} & \frac{\partial\lambda}{\partial y} & \frac{\partial\lambda}{\partial z}\end{pmatrix}\right\}, \qquad (16)$$

where $\tilde{G}_i^1 = \sum_{j=x,y,z} \frac{\partial \tilde{G}_i}{\partial a_j}$.

4.1. Round beam

In the case of a cylindrical symmetric ellipsoid $s_x = s_y = s_t$, the geometrical form factor reduces to the well known form factor of the round beam:

$$\begin{aligned} G_{xy}^{round}(\mu) &= G\left(\frac{a_x}{a_y}, \frac{a_x}{a_z}\right) = G\left(\frac{a_y}{a_x}, \frac{a_y}{a_z}\right) = \frac{3}{2(1-\mu)^{3/2}}\left[\sqrt{1-\mu} - \mu \ln\frac{1+\sqrt{1-\mu}}{\sqrt{\mu}}\right] \\ G_z^{round}(\mu) &= G\left(\frac{a_z}{a_x}, \frac{a_z}{a_y}\right) = \frac{3\mu}{(1-\mu)^{3/2}}\left[\frac{1}{2}\ln\frac{1+\sqrt{1-\mu}}{1-\sqrt{1-\mu}} - \sqrt{1-\mu}\right], \end{aligned} \tag{17}$$

where $\mu = a_x/a_z = a_y/a_z$. After some algebraic passages one can see that equation (14) reads $2 \cdot G_{xy}^{round}(\mu) + G_z^{round}(\mu) = 3$.

Geometrical form factors inside the round beam ($\mu = a_x/a_z = a_y/a_z$) as functions of the beam aspect ratio μ are shown in Fig.1

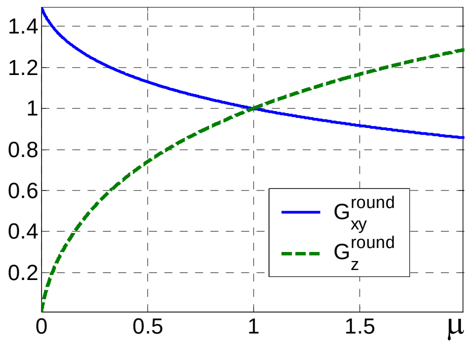

Figure 1. Geometrical form factors of the round beam

To determine field outside the driving Ensemble one should find λ from the equation:

$$\frac{(x^t - x^d)^2 + (y^t - y^d)^2}{s_t^2 + \lambda} + \frac{\gamma^2 (z^t - z^d)^2}{s_z^2 + \lambda} = 1, \tag{18}$$

and after the substitution of the equivalent ellipsoid aspect ratio $\mu = \frac{s_t^2 + \lambda}{s_z^2 + \lambda}$ into the geometric form factors (17) one can obtain the corresponding space charge force:

$$F_x^{d\to t}/e = \frac{q}{\gamma\sqrt{a_z}}\frac{x^t - x^d}{s_t^2+\lambda}G_{xy}^{round}(\mu),$$
$$F_y^{d\to t}/e = \frac{q}{\gamma\sqrt{a_z}}\frac{y^t - y^d}{s_t^2+\lambda}G_{xy}^{round}(\mu), \tag{19}$$
$$F_z^{d\to t}/e = \frac{q\gamma}{\sqrt{a_z}}\frac{z^t - z^d}{s_t^2+\lambda}G_{z}^{round}(\mu),$$

here it should be noticed that $\sqrt{a_z}$ implicitly contains γ. Matrix elements $\hat{\mathbf{F}}_{ij}^X$ can be obtained from (19) by direct analytical differentiation. An example of space charge field of the round beam is plotted in Fig 2.

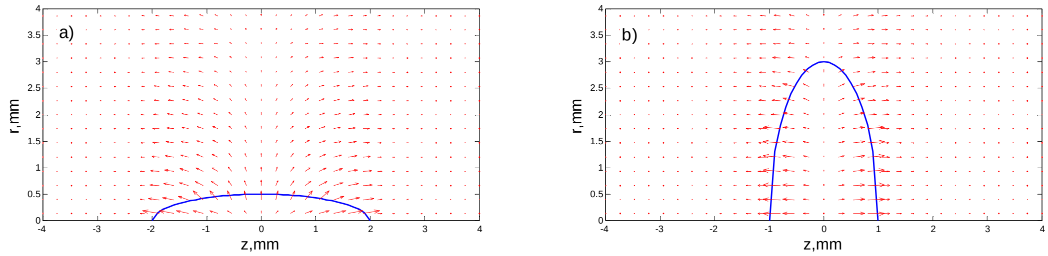

Figure 2. Space charge force of the round ellipsoidal beam (beam energy 5MeV): a) $XY_{rms} = 0.2mm, Z_{rms} = 0.9mm$; b) $XY_{rms} = 1.3mm, Z_{rms} = 0.4mm$ (pancake like bunch).

For the case of the small μ (long beam or high energy beam $\mu = \frac{5\sigma_x^2+\lambda}{5\gamma^2\sigma_z^2+\lambda} << 1$) space charge force for the round beam is given by

$$\frac{F_x^{d\to t}}{mc^2} = \frac{2I\sigma_z}{I_A\gamma\beta}\frac{x^t - x^d}{(s_t^2+\lambda)^{3/2}}\sqrt{\mu}\cdot\left[1+\mu\cdot\left(1-\ln\frac{2}{\sqrt{\mu}}\right)\right],$$
$$\frac{F_y^{d\to t}}{mc^2} = \frac{2I\sigma_z}{I_A\gamma\beta}\frac{y^t - y^d}{(s_t^2+\lambda)^{3/2}}\sqrt{\mu}\cdot\left[1+\mu\cdot\left(1-\ln\frac{2}{\sqrt{\mu}}\right)\right], \tag{20}$$
$$\frac{F_z^{d\to t}}{mc^2} = \frac{4I\sigma_z\gamma}{I_A\beta}\frac{z^t - z^d}{(s_t^2+\lambda)^{3/2}}\mu^{3/2}\cdot\left[\ln\frac{2}{\sqrt{\mu}}-1\right],$$

where $I_A = mc^3/e \approx 17kA$ is the Alfven current, $I = \frac{3}{4}q\beta c/\sigma_z$ is driving Ensemble current.

For the round coasting (i.e. unbunched) beam $\mu \to 0$, and these expressions take well known form):

$$\frac{F_r^{d\to t}}{mc^2} = \frac{2I}{I_A\gamma\beta}\begin{cases} r/s_t^2, & \text{if} \quad r < s_t \\ 1/r, & \text{if} \quad r < s_t \end{cases}, \tag{21}$$

where $r = \sqrt{\left(x^t - x^d\right)^2 + \left(y^t - y^d\right)^2}$.

4.2. Quasi-round beam

If the Ensemble transverse aspect ratio $\frac{a_x}{a_y} \approx 1 + \delta$, where $|\delta| << 1$, asymmetry corrections can be found analytically for correspondent $\mu_x = \frac{a_x}{a_z}$, $\mu_y = \frac{a_y}{a_z}$:

$$\begin{aligned} G_x(a_x, a_y, a_z) &\approx \left(1 - \delta \cdot \frac{4 - 7\mu_x}{8(1-\mu_x)}\right) \cdot G_{xy}^{round}(\mu_x) + \frac{3\delta}{8(1-\mu_x)} \\ G_y(a_x, a_y, a_z) &\approx \left(1 + \delta \cdot \frac{4 - 7\mu_y}{8(1-\mu_y)}\right) \cdot G_{xy}^{round}(\mu_y) - \frac{3\delta}{8(1-\mu_y)} \\ G_z(a_x, a_y, a_z) &= 3 - G_x(a_x, a_y, a_z) - G_y(a_x, a_y, a_z), \end{aligned} \tag{22}$$

where it has been assumed $\frac{a_y}{a_x} \approx 1 - \delta$. Terms with higher order of beam asymmetry ($\sim \delta^n$) can be similarly calculated.

4.3. Elliptic beam

For the case of elliptic coasting beam ($\mu_x \to 0, \mu_y \to 0$) transverse space charge force can be calculated rather easily

$$\begin{aligned} \frac{F_x^{d\to t}}{mc^2} &= \frac{4I}{I_A\gamma\beta}\frac{x^t - x^d}{\sqrt{a_x}\left(\sqrt{a_x} + \sqrt{a_y}\right)}, \\ \frac{F_y^{d\to t}}{mc^2} &= \frac{4I}{I_A\gamma\beta}\frac{y^t - y^d}{\sqrt{a_y}\left(\sqrt{a_x} + \sqrt{a_y}\right)}, \end{aligned} \tag{23}$$

for the points inside the driving Ensemble $\sqrt{a_x} = s_x$, $\sqrt{a_y} = s_y$, and expressions (23) coincide with well known formula for the space charge of the elliptic homogeneous distribution [7]. The formulae (23) are also valid for any external point (x, y), in this case $a_x = s_x^2 + \lambda$, $a_y = s_y^2 + \lambda$, where λ is a positive root of the equation ($z = 0$):

$$\frac{\left(x^t - x^d\right)^2}{s_x^2 + \lambda} + \frac{\left(y^t - y^d\right)^2}{s_y^2 + \lambda} = 1. \tag{24}$$

Fig.3 illustrates the field calculation using formulae (23), space charge force plotted as a function of transverse coordinates.

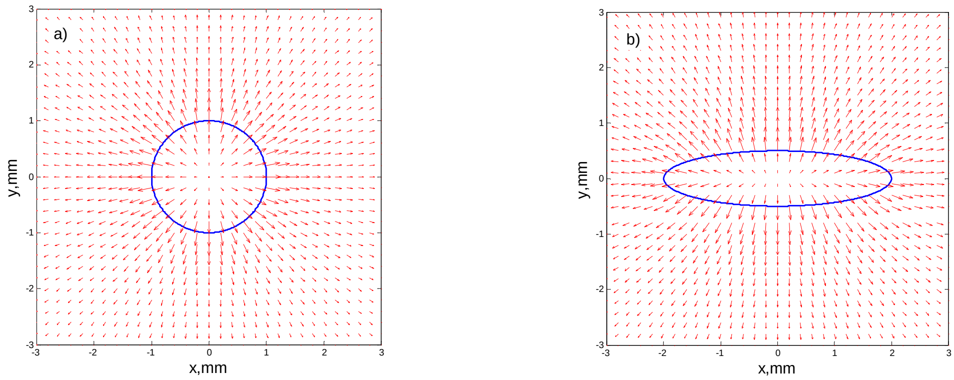

Figure 3. Space charge force of the elliptic beam: a) round beam with $X_{rms} = Y_{rms} = 1mm$; b) "flat" beam with $X_{rms} = 2mm$ $Y_{rms} = 0.5mm$

Besides the space charge field (22), the MEM needs also the matrix $\hat{\mathbf{F}}^X$ (6), for the elliptic beam the transverse matrix elements are given:

$$\hat{\mathbf{F}}^X_{tr} = \frac{I}{I_A\gamma\beta}\left\{\frac{4}{\sqrt{a_x}+\sqrt{a_y}}\begin{pmatrix}\frac{1}{\sqrt{a_x}} & 0\\ 0 & \frac{1}{\sqrt{a_y}}\end{pmatrix} - \frac{2}{\sqrt{a_x a_y}}\begin{pmatrix}\frac{\lambda'_x}{\sqrt{a_x}} & \frac{\lambda'_y}{\sqrt{a_x}}\\ \frac{\lambda'_x}{\sqrt{a_y}} & \frac{\lambda'_y}{\sqrt{a_y}}\end{pmatrix}\right\}, \tag{25}$$

the first term in this expression corresponds to the gradient inside the driving Ensemble ($\lambda = 0$), the second matrix determines a space charge field gradient outside, and besides the diagonal elements it contents also $X - Y$ coupling term. It should be noticed that inside the driving Ensemble the space charge force gradient causes growth of a test Ensemble rms size, whereas outside there is a test Ensemble contraction as a result of the negative space charge gradient (25).

5. Simulation of the Space Charge Dominated Beam

For the illustration of the proposed space charge algorithm a space charge dominated electron beam (1nC, 5MeV) in drift space has been simulated using Ensemble Model in comparison with conventional tracking code (ASTRA) [8]. Initial beam transverse phase space of the beam is shown in Fig 4, where equivalent phase space ellipses depict Ensemble parameters.

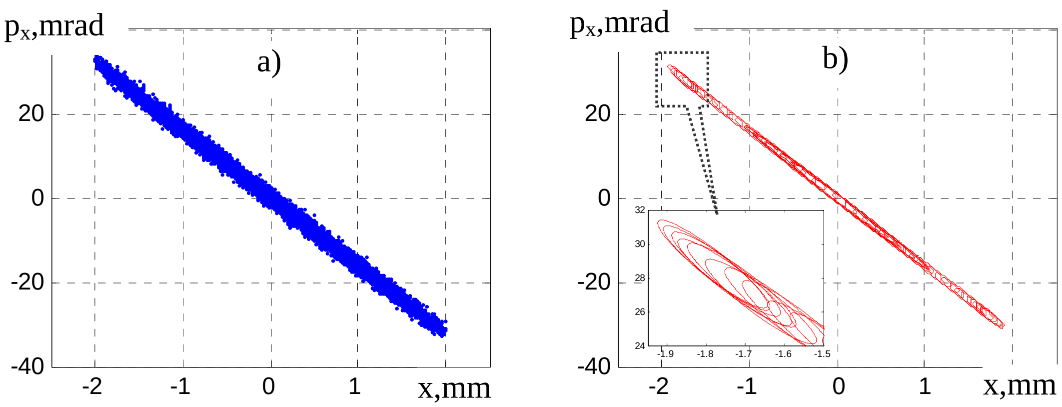

Figure 4. Initial transverse phase space (x, p_x): a) Conventional tracking code (ASTRA), 10000 macroparticles; b) Ensemble Model, 50 Ensembles.

Transverse RMS beam size, beam divergence and normilized beam emittance as functions of a flight time are shown in Fig 5 for the case without space charge. The agreement between conventional tracking code (ASTRA) and Ensemble Model (even using single Ensemble) is very good, whereas computation time is much smaller for the Ensemble Model (10 variables for SEM vs. 40000 ASTRA transverse particle coordinates).

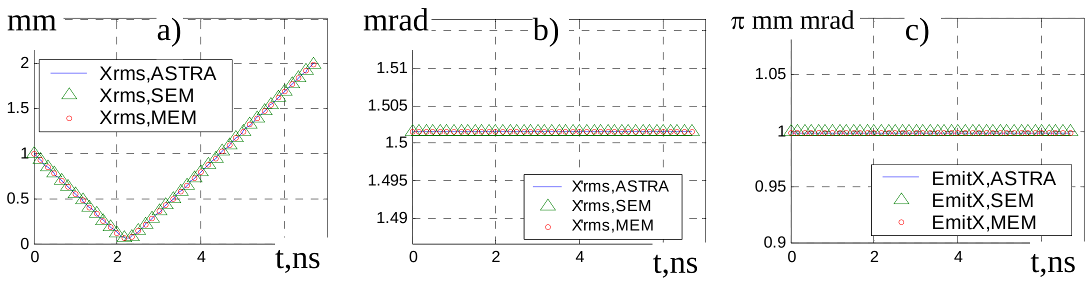

Figure 5. Simulations of the electron beam in a drift using conventional tracking code (ASTRA), SEM and MEM (50 Ensembles) without space charge. a) RMS beam size; b) RMS beam divergence; c) RMS beam emittance.

Corresponding dependencies for the case with space charge are shown in Fig. 6. The beam size as well as divergence can be simulated with SEM with rather good agreement, but for the emittance simulations MEM should be used.

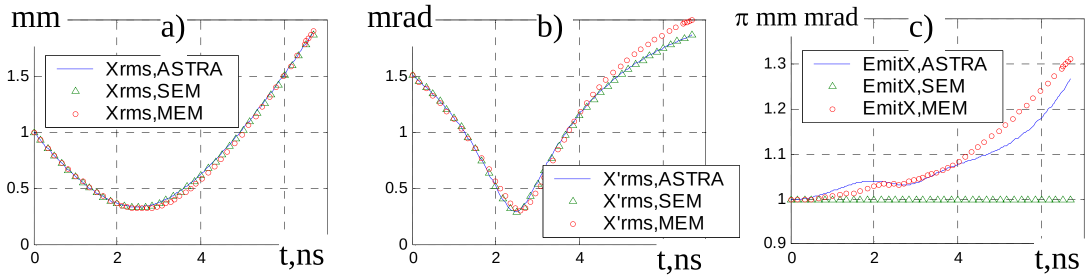

Figure 6. Simulations of the electron beam in a drift using conventional tracking code (ASTRA), SEM and MEM with space charge. a) RMS beam size; b) RMS beam divergence; c) RMS beam emittance.

Resulting phase spaces are shown in Fig. 7. It should be noted that a discrepancy in emittance caused mainly by non perfect interface between conventional macroparticles (10000 ASTRA particles) and Ensembles (50 Ensembles in MEM).

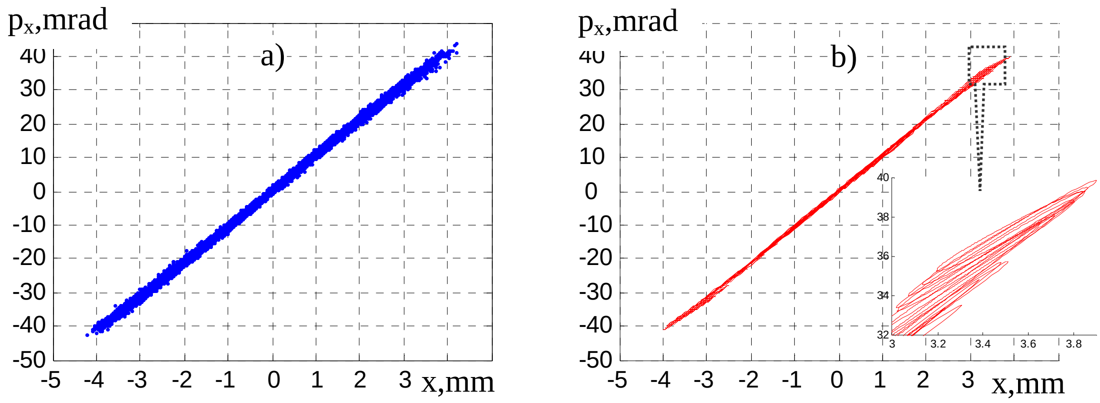

Figure 7. Final transverse phase space: a) Conventional tracking code (ASTRA); b) Ensemble Model.

As it can be seen from Fig. 6b, the RMS beam divergence decreases till the beam waist ($z \approx 2.5m$), whereas for the case without space charge it is constant (Fig. 5b). It should be figure out that this takes place even for the linear space charge algorithm (SEM). From the expression (25) for the space charge force one can possible to obtain the matrix element $\hat{\mathbf{F}}_{xx}^{X} = \dfrac{\kappa}{M_{xx}}$, where $\kappa = \dfrac{2}{5\beta\gamma}\dfrac{I}{I_A}$. Differential equations for transverse phase space take a form:

$$
\begin{aligned}
&\frac{dM_{p_x p_x}}{d\tau} = 2\kappa \cdot \frac{M_{xp_x}}{M_{xx}}, \\
&\frac{dM_{xp_x}}{d\tau} = \frac{M_{p_x p_x}}{\gamma_m} + \kappa, \\
&\frac{dM_{xx}}{d\tau} = \frac{2}{\gamma_m} \cdot M_{xp_x},
\end{aligned}
\qquad (26)
$$

Without solving this nonlinear system one can possible to obtain the following integral:

$$
M_{p_x p_x}(\tau) = M_{p_x p_x}(0) + \kappa\gamma_m \ln\frac{M_{xx}(\tau)}{M_{xx}(0)}. \qquad (27)
$$

Hence, applying a space charge force under definite (negative) phase space correlation the logarithmic term in (27) is negative (before the focus $M_{xx}(\tau) < M_{xx}(0)$) and rms beam divergence decreases as a result of the space charge effect.

6. Conclusions

Space charge algorithm for the Multi Ensemble Model (MEM) based on the 3-axis homogeneously charged ellipsoid has been developed. Analytical expressions for the Lorenz force and its gradients have been obtained. MEM simulations of the nonlinear effects in transverse phase space of the space charge dominated beam showed good agreement with conventional tracking code, demonstrating advantage in variable number and computational efficiency.

References

[1] A. Novokhatski, T. Weiland, PAC'99, New York, March 1999

[2] M. Krassilnkov et al., ICAP 2000, Darmstadt, September 2000

[3] M. Krassilnikov, T. Weiland, ICAP'02, East Lansing, USA, October 2002

[4] M. Comunian et al. Phys.Rev. Spec.Topics- Acc. and Beams, V.4, 124201 (2001)

[5] I.M. Kapchinskiy “Theory of Resonance Linear Acceler.”, Harw.Ac.Pub., 1985.

[6] W.J. Sternberg, T.L. Smith, “The Theory of Potential and Spherical Harmonics”, Harmonics, University of Toronto Press, 1952.

[7] H. Wiedemann, “Particle accelerator physics:Basic principles and linear beam dynamics”, Berlin Springer, 1993

[8] ASTRA User Manual, http://www.desy.de/~mpyflo/Astra_documentation/