Initial and boundary value problems for the Vlasov equation

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Kinetic approach to plasma

- particle distribution function for N particles:

$$f_N(t, \overrightarrow{r_1}, \overrightarrow{r_2}, \dots, \overrightarrow{r_N}, \overrightarrow{p_1}, \overrightarrow{p_2}, \dots, \overrightarrow{p_N}) \qquad (1.1)$$

- for noninteracting particles (collisionless plasma)

$$f_N(t, \overrightarrow{r_1}, \overrightarrow{r_2}, \dots, \overrightarrow{r_N}, \overrightarrow{p_1}, \overrightarrow{p_2}, \dots, \overrightarrow{p_N}) = \prod_{i=1}^{N} f(t, \overrightarrow{r_i}, \overrightarrow{p_i}) \qquad (1.2)$$

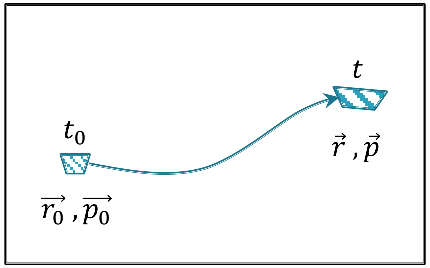
- probability that the particle is within the volume $d\vec{r}\, d\vec{p}$ around the point $\vec{r}, \vec{p}$ of the phase space at the t time moment:

$$f(t, \vec{r}, \vec{p})\, d\vec{r}\, d\vec{p} \qquad (1.3)$$

- normalization → NoP:

$$\int f(t, \vec{r}, \vec{p})\, d\vec{r}\, d\vec{p} = N \qquad (1.4.)$$

Kinetic equation for $f(t,\overrightarrow{r_i},\overrightarrow{p_i})$



$$\frac{d\vec{r}}{dt} = \vec{v}\,, \quad \frac{d\vec{p}}{dt} = \vec{F} = e\left(\vec{E} + \left[\vec{v} \times \vec{B}\right]\right) \qquad (1.5)$$

$$\vec{r}(t_0) = \overrightarrow{r_0}\,, \qquad \vec{p}(t_0) = \overrightarrow{p_0} \qquad (1.6)$$

$$d\overrightarrow{r_0} \cdot d\overrightarrow{p_0} \to d\vec{r} \cdot d\vec{p}, \quad f(t_0,\overrightarrow{r_0},\overrightarrow{p_0}) \to f(t,\vec{r}(t),\vec{p}(t)) \qquad (1.7)$$

- Assuming the invariance of the particle number (no ionization, no recombination, no collisions) a full particle number in the phase space volume is constant:

$$f(t,\vec{r}(t),\vec{p}(t))\, d\vec{r}\, d\vec{p} = f(t_0,\overrightarrow{r_0},\overrightarrow{p_0})\, d\overrightarrow{r_0}\, d\overrightarrow{p_0} = const \qquad (1.8)$$

- According to the Liouville's theorem the phase space volume is preserved:

$$d\overrightarrow{r_0} \cdot d\overrightarrow{p_0} = 1 \cdot d\vec{r} \cdot d\vec{p} \qquad (1.9)$$

➔the particle distribution function along the phase trajectory is constant:

$$f(t,\vec{r}(t),\vec{p}(t)) = const \qquad (1.10)$$

Kinetic equation for $f(t, \vec{r_i}, \vec{p_i})$

$$\frac{df(t,\vec{r}(t),\vec{p}(t))}{dt} = \frac{\partial f}{\partial t} + \frac{\partial f}{\partial \vec{r}}\frac{d\vec{r}(t)}{dt} + \frac{\partial f}{\partial \vec{p}}\frac{d\vec{p}(t)}{dt} = 0 \qquad (1.11)$$

- Combining with (1.5)

$$\frac{d\vec{r}}{dt} = \vec{v}, \quad \frac{d\vec{p}}{dt} = \vec{F} = e\left(\vec{E} + [\vec{v}\times\vec{B}]\right) \qquad (1.5)$$

- **Vlasov equation:**

$$\frac{\partial f}{\partial t} + \vec{v}\frac{\partial f}{\partial \vec{r}} + e\left(\vec{E} + [\vec{v}\times\vec{B}]\right)\frac{\partial f}{\partial \vec{p}} = 0 \qquad (1.12)$$

- +Maxwell equations for the electromagnetic fields:

$$\nabla\vec{E} = \frac{\rho}{\varepsilon_0}, \quad \nabla\vec{B} = 0, \quad [\nabla\times\vec{E}] = -\frac{\partial\vec{B}}{\partial t}, \quad [\nabla\times\vec{B}] = \mu_0\vec{j} + \frac{1}{c^2}\frac{\partial\vec{E}}{\partial t}, \qquad (1.13)$$

  - charge and current densities are calculated as

$$\rho(t,\vec{r}) = e\int f(t,\vec{r},\vec{p})d\vec{p}, \qquad \vec{j} = e\int \vec{v} f(t,\vec{r},\vec{p})d\vec{p} \qquad (1.14)$$

  - particle density $n(t,\vec{r})$:

$$\int f(t,\vec{r},\vec{p})d\vec{p} = n(t,\vec{r}), \qquad (1.15)$$

$$\int n(t,\vec{r})d\vec{r} = N \qquad (1.16)$$

NB: only variables $t, \vec{r}, \vec{p}$ in (1.12) are independent $\vec{v} = c\frac{\vec{p}}{\sqrt{m^2c^2+p^2}}$ (1.17)

$c = \frac{1}{\sqrt{\varepsilon_0\mu_0}} = 299792458\frac{m}{s}$

$\varepsilon_0 = 8.854\cdot 10^{-12}\frac{A\,s}{V\,m}$

$\mu_0 = 4\pi\cdot 10^{-7}\frac{V\,s}{A\,m}$

(1.12)+(1.13)+(1.14)=
!complete
!rigorous
!full physics

Vlasov Equation: solution?

$$\frac{\partial f}{\partial t} + \vec{v}\frac{\partial f}{\partial \vec{r}} + e\left(\vec{E} + \left[\vec{v} \times \vec{B}\right]\right)\frac{\partial f}{\partial \vec{p}} = 0$$

- Two main approaches:
  - Initial value = initial space distribution → evolution → time *t* as independent variable.
  - Boundary value = boundary conditions → injection (emission) → e.g. *z* as independent variable.

- Both problems can be formulated as a mathematical Cauchy problem for the solution of a partial differential equation (PDE) that satisfies corresponding conditions which are given on a hypersurface in the domain.

M: General Cauchy problem for linear PDE

- The Cauchy problem for the first order linear homogeneous PDE:

$$W_1 \frac{\partial f}{\partial u_1} + W_2 \frac{\partial f}{\partial u_2} + \cdots W_n \frac{\partial f}{\partial u_n} + W_{n+1} \frac{\partial f}{\partial u_{n+1}} = 0, \qquad (2.1)$$

where $W_i = W_i(u_1, u_2, \dots, u_n, u_{n+1})$ are given functions of $n+1$ independent variables $(u_1, u_2, \dots, u_n, u_{n+1})$.

- additional condition for the fixed selected variable x:

$$f(0, u_2, \dots, u_n, u_{n+1}) = R(u_2, u_3, \dots, u_n, u_{n+1}) \qquad (2.2)$$

- Let us:
  - fix the first independent variable $u_1 = x$ as a selected ("evolution") variable x.
  - denote $Y = W_1$
  - introduce new vectors (with dimension of n) as

$$\vec{q} = \{u_2, u_3, \dots, u_n, u_{n+1}\}, \quad \vec{G} = \{W_2, W_3, \dots, W_n, W_{n+1}\} = \vec{G}(x, \vec{q}) \qquad (2.3)$$

- The equation (2.1) can be rewritten as

$$\boldsymbol{Y \frac{\partial f}{\partial x} + \vec{G} \cdot \frac{\partial f}{\partial \vec{q}} = 0} \qquad (2.4)$$

- The condition (2.2) takes a form:

$$\boldsymbol{f(0, \vec{q}) = R(\vec{q})} \qquad (2.5)$$

M: General Cauchy problem solution

$$Y\frac{\partial f}{\partial x}+\vec{G}\cdot\frac{\partial f}{\partial\vec{q}}=0 \qquad (2.4)$$

Three steps:

(I) Form the characteristic equations:

$$\frac{dx}{Y}=\frac{d\vec{q}}{\vec{G}}=\frac{dq_1}{G_1}=\frac{dq_2}{G_2}=\cdots=\frac{dq_n}{G_n}, \qquad (2.6)$$

This system can be rewritten in vector form as system of ordinary differential equations (ODE):

$$\frac{d\vec{q}}{dx}=\frac{\vec{G}}{Y} \qquad (2.7)$$

(II) Obtain the first n integrals of the system (2.7):

$$\vec{\Psi}(x,\vec{q})=\vec{C}, \qquad (2.8)$$

where the vector $\vec{C}$ is an arbitrary constant vector. Then the equation (2.8) has to be resolved w.r.t. the unknown vector $\vec{q}$:

$$\vec{q}=\vec{\mathrm{Q}}(x,\vec{C}), \qquad (2.9)$$

(III) The first integral (2.8) is to substituted in an arbitrary differentiable function $\Phi(\vec{C})$

$$f(x,\vec{q})=\Phi(\vec{C})=\Phi\left(\vec{\Psi}(x,\vec{q})\right) \qquad (2.10)$$

M: Cauchy problem solution (practical scheme)

- The solution of the Cauchy problem can be obtained using the same scheme but applying additional conditions:

$$\vec{q}(\mathrm{x} = 0) = \vec{q}_0, \tag{2.11}$$

- The solution of the characteristic system can be rewritten using (2.11), where the arbitrary constant vector is replaced with $\vec{q}_0$:

$$\vec{q} = \vec{\mathrm{Q}}(x, \vec{q}_0) \tag{2.12}$$

- Expressing the initial vector $\vec{q}_0$ from (2.12) the first integral of the characteristic system can be obtained:

$$\vec{q}_0 = \vec{Q}_0(x, \vec{q}) \tag{2.13}$$

- Substitution of the (2.13) in in an arbitrary differentiable function Φ results in in a general solution:

$$f(x, \vec{q}) = \Phi\left(\vec{Q}_0(x, \vec{q})\right) \tag{2.14}$$

M: Cauchy problem solution (practical)

- For the final solution of the Cauchy problem the arbitrary function Φ has to be found. Using the additional condition one obtains:

$$f(0,\vec{q}) = \Phi\left(\vec{Q}_0(0,\vec{q})\right) = R(\vec{q}) \tag{2.15}$$

- The function $\vec{q}_0 = \vec{Q}_0(x,\vec{q})$ is constant along the vector line – characteristics – by definition. That is why the solution of the solution of the Cauchy problem is also conserved along the characteristics. This means that arbitrary function Φ can be obtained as:

$$\Phi(\vec{q}) = R(\vec{q}) \tag{2.16}$$

- Finally the solution of the Cauchy problem is given by equation:

$$f(x,\vec{q}) = R\left(\vec{Q}_0(x,\vec{q})\right) \tag{2.17}$$

$$\vec{q} = \vec{Q}(x,\vec{q}_0) \Rightarrow \vec{q}_0 = \vec{Q}_0(x,\vec{q}) \Rightarrow R(\vec{q}_0) \Rightarrow f(x,\vec{q})$$

1D Vlasov equation

- In the 1D case the distribution function $f(t,\vec{r},\vec{p})$ does not depend on transverse coordinated x and y.
- The 1D plasma is considered in the presence of the longitudinal Lorenz force $\vec{F} = \{0,0,F_z\}$.
- This could be, for example,
  - the case of electron beam start in the cathode vicinity of the rf gun: $F_z = eE_0 \sin(\omega t + \varphi_0)$.
  - the case of electron beam acceleration in the plasma wake field.

1D Vlasov equation: initial value problem

- The 1D Vlasov equation with initial condition are written as

$$\frac{\partial f}{\partial t} + v_z \frac{\partial f}{\partial z} + F_z \frac{\partial f}{\partial p_z} = 0 \qquad (3.1)$$

$$f(t=0, z, p_z) = f_0(z, p_z) \qquad (3.2)$$

- The equations for the characteristics are straightforward:

$$\frac{dt}{1} = \frac{dz}{v_z} = \frac{dp_z}{F_z} \qquad (3.3)$$

- or in the ODE form (t is fixed coordinate)

$$\frac{dz}{dt} = v_z, \quad \frac{dp_z}{dt} = F_z \qquad (3.4)$$

- Initial conditions:

$$z(t=0) = z_0, \qquad p_z\,(t=0) = p_{z0} \qquad (3.5)$$

1D initial value problem: case $F_z = F_0 = const$

- Solution of characteristics:

$$p_z = p_{z0} + F_0 t, \quad z = z_0 + \frac{c}{F_0}\left[\sqrt{m^2c^2 + (p_{z0} + F_0 t)^2} - \sqrt{m^2c^2 + {p_{z0}}^2}\right] \quad (3.6)$$

- Following the above mentioned scheme we have to express p_{z0} and z_0 from this solution:

$$p_{z0} = p_z - F_0 t, \quad z_0 = z + \frac{c}{F_0}\left[\sqrt{m^2c^2 + (p_z - F_0 t)^2} - \sqrt{m^2c^2 + {p_z}^2}\right] \quad (3.7)$$

- Substituting (3.7) into (3.2) one obtains the solution of the initial value problem (3.1)–(3.2) for the constant field:

$$f(t, z, p_z) = f_0(z_0, p_{z0}) = f_0\left(z + \frac{c}{F_0}\left[\sqrt{m^2c^2 + (p_z - F_0 t)^2} - \sqrt{m^2c^2 + {p_z}^2}\right], p_z - F_0 t\right) \quad (3.8)$$

1D initial value problem: case $F_z = F_0 = const$

- As an example a start of the bunched cold electron beam can be considered. In this case the initial particle distribution function can be factorized and represented as:

$$f_0(z, p_z) = G(z) \cdot \delta(p_z) \qquad (3.9)$$

here $G(z)$ is a longitudinal bunch distribution (e.g. Gaussian or a flattop profile), $\delta(p_z)$ is a Dirac delta function which assumes a start of the cold beam (zero momentum with zero momentum spread). Integration in p_z results in the following solution – charge density distribution function:

$$\rho(t, z) = G\left(z + \frac{c}{F_0}\left[mc - \sqrt{m^2c^2 + (F_0 t)^2}\right]\right) \qquad (3.10)$$

- The nonrelativistic approximation of (3.10) can be easily obtained:

$$\rho_{nr}(t, z) = G\left(z - \frac{F_0 t^2}{2m}\right) \qquad (3.11)$$

1D initial value problem: case $F_z \sim \sin(\omega t)$

- The external force can be represented as

$$F_z(t) = eE_0 \sin(\omega t + \varphi_0), \tag{3.12}$$

where E_0 and φ_0 are amplitude and the initial phase of the accelerating field. For simplicity let us consider the nonrelativistic case.

- The solution of the characteristic is given by

$$p_z = p_{z0} + \alpha m\omega \cdot [\cos\varphi_0 - \cos(\omega t + \varphi_0)] \tag{3.13a}$$

$$z = z_0 + \frac{p_{z0}}{m} t + \alpha \cdot [\omega t \cos\varphi_0 - \sin(\omega t + \varphi_0) + \sin\varphi_0] \tag{3.13b}$$

- The expressions for p_{z0} and z_0 take a form:

$$p_{z0} = p_z + \alpha m\omega \cdot [\cos(\omega t + \varphi_0) - \cos\varphi_0] \tag{3.14a}$$

$$z_0 = z - \frac{p_z}{m} t + \alpha \cdot [\sin(\omega t + \varphi_0) - \sin\varphi_0 - \omega t \cos(\omega t + \varphi_0)] \tag{3.14b}$$

Here $\alpha = \frac{eE_0}{m\omega^2}$ is normalized amplitude of the field.

- The solution of the initial value problem in this case is

$$f(t, z, p_z) = f_0\left(\begin{matrix} z - \frac{p_z}{m} t + \alpha \cdot [\sin(\omega t + \varphi_0) - \sin\varphi_0 - \omega t \cos(\omega t + \varphi_0)], \\ p_z + \alpha m\omega \cdot [\cos(\omega t + \varphi_0) - \cos\varphi_0] \end{matrix}\right) \tag{3.15}$$

- Assuming the initial distribution like (3.9) yields the particle distribution function:

$$\rho(t, z) = G(\, z + \alpha \cdot [\sin(\omega t + \varphi_0) - \sin\varphi_0 - \omega t \cos\varphi_0]) \tag{3.16}$$

1D Initial problem: numerical example

Parameters:

- $E_0 = 60\frac{MV}{m}$, $\omega = 2\pi \cdot 1.3GHz$, which corresponds to $\alpha = 0.158m$.
- Gaussian initial distribution with 2 mm rms bunch length
- 3 initial (launch) phases $\varphi_0 = 10; 40; 70deg$

- (upper row) – static nonrelativistic solution (3.11):

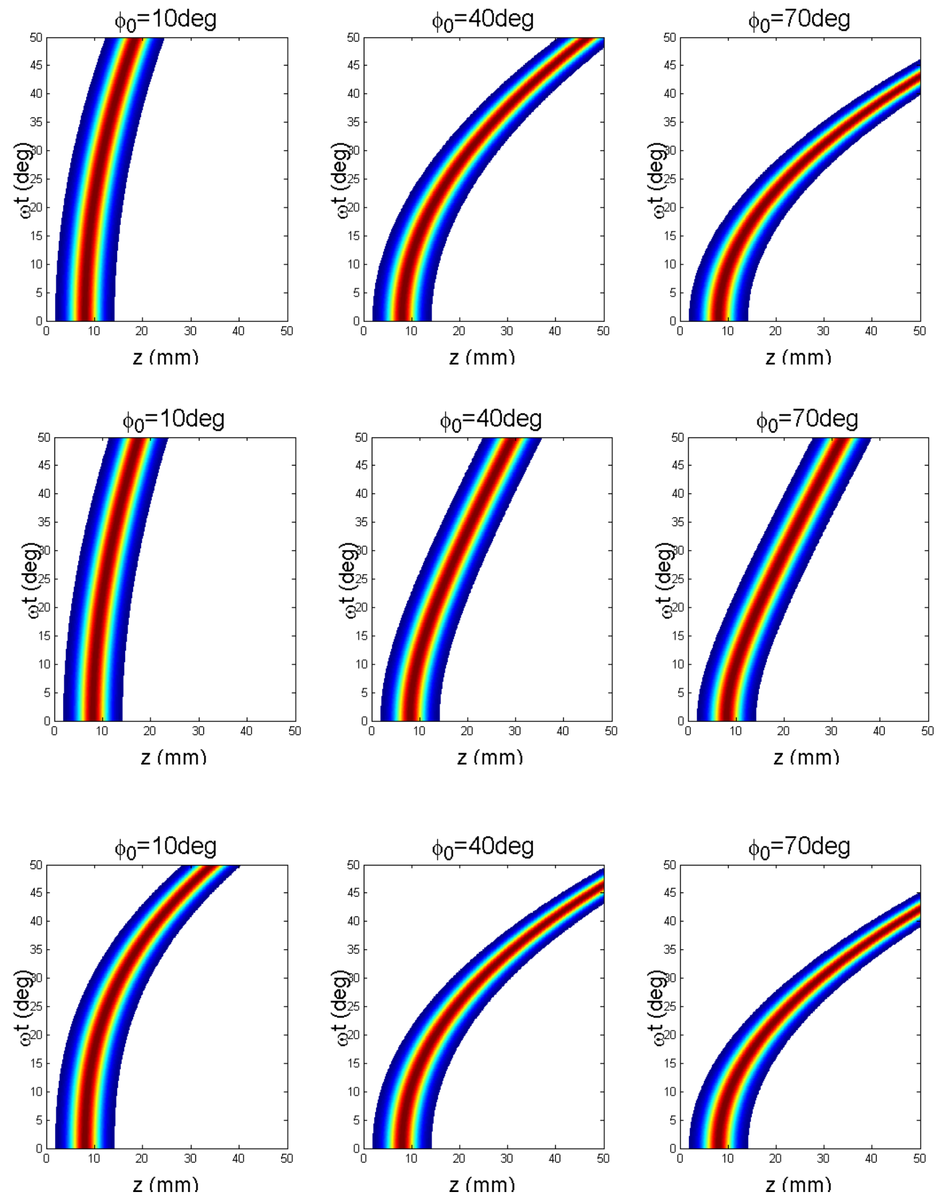
$$\rho_{nr}(t,z) = G\left(z - \frac{eE_0 \sin\varphi_0 \, t^2}{2m}\right)$$

- (middle row) – static relativistic solution (3.10) ($F_0 \to eE_0 \sin\varphi_0$):

$$\rho(t,z) = G\left(z + \frac{c}{F_0}\left[mc - \sqrt{m^2c^2 + (F_0 t)^2}\right]\right)$$

- (bottom row) – time dependent nonrelativistic solution (3.16):

$$\rho(t,z) = G(z + \alpha \times \times [\sin(\omega t + \varphi_0) - \sin\varphi_0 - \omega t \cos\varphi_0])$$



1D Vlasov equation: Boundary value problem

- the problem of the plasma (beam) injection through the *z=0* plane.

$$\frac{\partial f}{\partial t} + v_z \frac{\partial f}{\partial z} + F_z \frac{\partial f}{\partial p_z} = 0 \qquad (3.17)$$

$$f(t, z = 0, p_z) = f_0(t, p_z) \qquad (3.18)$$

- now the ***z*** coordinate serves as a fixed ('evolution") variable.
- characteristic system (3.3) remains but it should be resolved now in a different way:

$$\frac{dt}{dz} = \frac{1}{v_z}, \qquad \frac{dp_z}{dz} = \frac{F_z}{v_z} \qquad (3.19)$$

- this system has to be completed by boundary conditions

$$t(z = 0) = t_0, \qquad p_z\,(z = 0) = p_{z0} \qquad (3.20)$$

1D boundary value problem: case $F_z = F_0 = const$

- For the simplicity let us consider a nonrelativistic case $p_z = mv_z$ with a constant Lorenz force $F_z = F_0 = const$. Under these assumptions the solution of the characteristic system (3.19)–(3.20) can be written as

$$p_z = \sqrt{p_{z0}^2 + 2mF_0 z}, \quad t = t_0 + \frac{1}{F_0}\left[\sqrt{p_{z0}^2 + 2mF_0 z} - p_{z0}\right] \qquad (3.21)$$

- Following the same scheme as above:

$$p_{z0} = \sqrt{p_z^2 - 2mF_0 z}, \quad t_0 = t + \frac{1}{F_0}\left[\sqrt{p_z^2 - 2mF_0 z} - p_z\right] \qquad (3.21)$$

- The solution of the boundary value problem in this case takes a form:

$$f(t,z,p_z) = f_0(t_0,p_{z0}) = f_0\left(t + \frac{1}{F_0}\left[\sqrt{p_z^2 - 2mF_0 z} - p_z\right], \sqrt{p_z^2 - 2mF_0 z}\right) \quad (3.22)$$

- Assuming that cold plasma (beam) injected through the z=0 plane has temporal profile $G(t)$:

$$f(t,z=0,p_z) = f_0(t,p_z) = G(t) \cdot \delta(p_z) \qquad (3.23)$$

- and applying an integration in p_z one obtains for the particle density evolution:

$$\rho(t,z) = G\left(t - \sqrt{\frac{2mz}{F_0}}\right) \qquad (3.24)$$

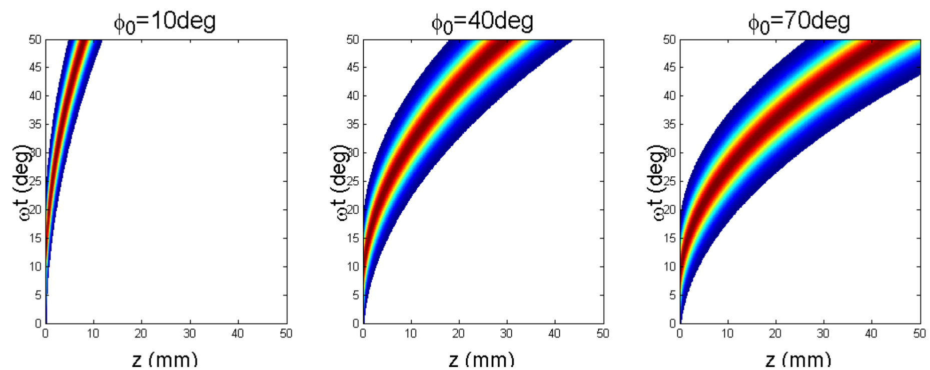
1D boundary value problem: case $F = F_0 = const$

- For the nonrelativistic (but still static) case:

$$\rho(t,z) = G\left(t - \sqrt{\frac{2mz}{F_0}}\right) \qquad (3.24)$$

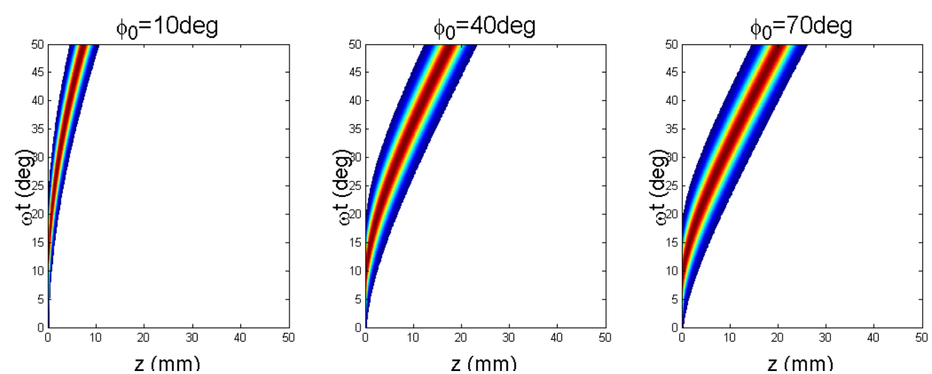
Parameters:

- $E_0 = 60\frac{MV}{m}$, $\omega = 2\pi \cdot 1.3GHz$, which corresponds to $\alpha = 0.158m$.
- Gaussian temporal profile with 6.6 ps rms duration
- 3 initial (launch) phases $\varphi_0 = 10; 40; 70deg$



- For the relativistic (but still static) case (➔ *HW*):

$$\rho(t,z) = G\left(t - \frac{1}{F_0}\sqrt{\frac{F_0 z}{c}\cdot\left(2mc + \frac{F_0 z}{c}\right)}\right) \qquad (3.25)$$



Summary

- Collisionless plasma → kinetic aproach
- Kinetic Vlasov equation
- Cauchy problem for PDE
- Cauchy problem for Vlasov equation:
  - Initial value problem
  - Boundary value problem

BUT! This method is rather restricted:

strong nonlinearity → fields (external + self)

→integration over initial (boundary) conditions

$$f(t,z,p_z) = \int dt_0 dp_{z0}\, f_0(t_0,p_{z0})\delta[t - T(t,z,p_z,t_0,p_{z0})]\delta[p_z - P_z(t,z,p_z,t_0,p_{z0})]$$