Electrodynamics of media with time and space dispersion

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L#3

Maxwell Equations

$$\nabla\vec{E} = \frac{\rho}{\varepsilon_0} + \frac{\rho_0}{\varepsilon_0}$$

$$\nabla\vec{B} = 0$$

$$[\nabla \times \vec{E}] = -\frac{\partial\vec{B}}{\partial t} \qquad (3.1)$$

$$[\nabla \times \vec{B}] = \frac{1}{c^2}\frac{\partial\vec{E}}{\partial t} + \mu_0\vec{j} + \mu_0\vec{j_0}$$

$\rho_0; \vec{j_0}$ → external field sources

$$c = \frac{1}{\sqrt{\varepsilon_0\mu_0}} = 299792458\,\frac{m}{s}$$

$$\varepsilon_0 = 8.854 \cdot 10^{-12}\,\frac{A\,s}{V\,m}$$

$$\mu_0 = 4\pi \cdot 10^{-7}\,\frac{V\,s}{A\,m}$$

$$\vec{F} = e\left(\vec{E} + [\vec{v} \times \vec{B}]\right) \qquad (3.2)$$

Maxwell Equations

$$\nabla \vec{E} = \frac{\rho}{\varepsilon_0} + \frac{\rho_0}{\varepsilon_0}$$

$\nabla \cdot$

$$[\nabla \times \vec{B}] = \frac{1}{c^2}\frac{\partial \vec{E}}{\partial t} + \mu_0 \vec{j} + \mu_0 \overrightarrow{j_0}$$

$$0 = \frac{1}{c^2}\frac{\partial}{\partial t}\nabla \vec{E} + \mu_0 \nabla \vec{j} + \mu_0 \nabla \overrightarrow{j_0}$$

$$c = \frac{1}{\sqrt{\varepsilon_0 \mu_0}}$$

$$\frac{\partial}{\partial t}(\rho + \rho_0) + \nabla\left(\vec{j} + \overrightarrow{j_0}\right) = 0 \qquad (3.3)$$

Equation of continuity

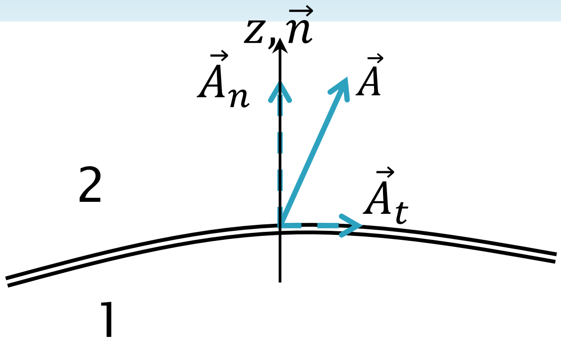
Electric displacement field

$$\vec{D}(t,\vec{r}) = \varepsilon_0\vec{E}(t,\vec{r}) + \int_{-\infty}^{t} dt'\vec{j}\left(t',\vec{r}\right) \qquad (3.4)$$

$$\begin{gathered}
\nabla\vec{D} = \rho_0 \\
\nabla\vec{B} = 0 \\
\left[\nabla\times\vec{E}\right] = -\frac{\partial\vec{B}}{\partial t} \\
\left[\nabla\times\vec{B}\right] = \frac{1}{c^2\varepsilon_0}\frac{\partial\vec{D}}{\partial t} + \mu_0\overrightarrow{j_0}
\end{gathered} \qquad (3.5)$$

$$???\,\vec{D} \leftrightarrow \vec{E}$$

Boundary conditions



$$\nabla \vec{B} = 0 \Rightarrow \vec{B}_{2n} - \vec{B}_{1n} = 0$$

$$[\nabla \times \vec{E}] = -\frac{\partial \vec{B}}{\partial t} \Rightarrow \vec{E}_{2t} - \vec{E}_{1t} = 0$$

$$\nabla \vec{D} = \rho_0 \Rightarrow \vec{D}_{2n} - \vec{D}_{1n} = \sigma_0 - \underbrace{\int_1^2 \nabla \left[\vec{n} \times [\vec{D} \times \vec{n}]\right] dz}$$

$= \sigma \rightarrow$ surface density of the induced charge

$$[\nabla \times \vec{B}] = \frac{1}{c^2 \varepsilon_0} \frac{\partial \vec{D}}{\partial t} + \mu_0 \vec{j_0}$$

$$[\overrightarrow{n \times} (\vec{B}_2 - \vec{B}_1)] = \vec{B}_{2t} - \vec{B}_{1t} = \\ = \frac{1}{c^2 \varepsilon_0} \left(i_0 - \int_1^2 \vec{j} dz\right)$$

$$i_0 = \int_1^2 \vec{j}_0 dz$$

surface current density

Material Equations

??? $\vec{D} \leftrightarrow \vec{E}$ or $\vec{j} \leftrightarrow \vec{E}$???

In general case: $\vec{D}(t,\vec{r}) = \Phi\left(\vec{E}(t,\vec{r})\right); \vec{j}(t,\vec{r}) = \Psi\left(\vec{E}(t,\vec{r})\right)$

Material equations for **linear** electrodynamics

$$D_i(t,\vec{r}) = \varepsilon_0 \int_{-\infty}^{t} dt_1 \int d\vec{r}_1\, \hat{\varepsilon}_{ij}(\vec{r},\vec{r}_1,t,t_1) E_j(\vec{r}_1,t_1)$$

$$j_i(t,\vec{r}) = \int_{-\infty}^{t} dt_1 \int d\vec{r}_1\, \hat{\sigma}_{ij}(\vec{r},\vec{r}_1,t,t_1) E_j(\vec{r}_1,t_1)$$

Influence functions

Homogeneous Media

$$D_i(t,\vec{r}) = \varepsilon_0 \int_{-\infty}^{t} dt_1 \int d\vec{r}_1 \, \hat{\varepsilon}_{ij}(\vec{r},\vec{r}_1,t,t_1) E_j(\vec{r}_1,t_1)$$

$$j_i(t,\vec{r}) = \int_{-\infty}^{t} dt_1 \int d\vec{r}_1 \, \hat{\sigma}_{ij}(\vec{r},\vec{r}_1,t,t_1) E_j(\vec{r}_1,t_1)$$

Homogeneity in ***t*** and ***space***

$$D_i(t,\vec{r}) = \varepsilon_0 \int_{-\infty}^{t} dt_1 \int d\vec{r}_1 \, \hat{\varepsilon}_{ij}(\vec{r}-\vec{r}_1,t-t_1) E_j(\vec{r}_1,t_1)$$

$$j_i(t,\vec{r}) = \int_{-\infty}^{t} dt_1 \int d\vec{r}_1 \, \hat{\sigma}_{ij}(\vec{r}-\vec{r}_1,t-t_1) E_j(\vec{r}_1,t_1)$$

Fourier Transform

$$D_i(t,\vec{r}) = \varepsilon_0 \int_{-\infty}^{t} dt_1 \int d\vec{r}_1\, \hat{\varepsilon}_{ij}(\vec{r}-\vec{r}_1, t-t_1) E_j(\vec{r}_1, t_1)$$

$$j_i(t,\vec{r}) = \int_{-\infty}^{t} dt_1 \int d\vec{r}_1\, \hat{\sigma}_{ij}(\vec{r}-\vec{r}_1, t-t_1) E_j(\vec{r}_1, t_1)$$

$$\vec{E}(t,\vec{r}) = \int_{-\infty}^{\infty} d\omega \int d\vec{\boldsymbol{k}}\ \vec{E}(\omega,\vec{k}) e^{-i\omega t + i\vec{k}\vec{r}}$$

$$\tilde{\vec{r}} = \vec{r} - \vec{r}_1$$

$$\tilde{t} = t - t_1$$

$$D_i(\omega,\vec{k}) = \varepsilon_0 \int_0^{\infty} d\tilde{t} \int d\tilde{\vec{r}}\, \hat{\varepsilon}_{ij}(\tilde{\vec{r}},\tilde{t}) e^{i\omega\tilde{t} - i\vec{k}\tilde{\vec{r}}} E_j(\omega,\vec{k}) = \varepsilon_0 \varepsilon_{ij}(\omega,\vec{k}) E_j(\omega,\vec{k})$$

$$j_i(\omega,\vec{k}) = \int_0^{\infty} d\tilde{t} \int d\tilde{\vec{r}}\, \hat{j}_{ij}(\tilde{\vec{r}},\tilde{t}) e^{i\omega\tilde{t} - i\vec{k}\tilde{\vec{r}}} E_j(\omega,\vec{k}) = \sigma_{ij}(\omega,\vec{k}) E_j(\omega,\vec{k})$$

Complex tensors $\varepsilon_{ij}(\omega,\vec{k})$ and $\sigma_{ij}(\omega,\vec{k})$

$$D_i(\omega,\vec{k}) = \varepsilon_0 \int_0^\infty d\tilde{t} \int d\tilde{\vec{r}}\, \hat{\varepsilon}_{ij}(\tilde{\vec{r}},\tilde{t}) e^{i\omega\tilde{t}-i\vec{k}\tilde{\vec{r}}} E_j(\omega,\vec{k}) = \varepsilon_0 \varepsilon_{ij}(\omega,\vec{k}) E_j(\omega,\vec{k})$$

$$j_i(\omega,\vec{k}) = \int_0^\infty d\tilde{t} \int d\tilde{\vec{r}}\, \hat{j}_{ij}(\tilde{\vec{r}},\tilde{t}) e^{i\omega\tilde{t}-i\vec{k}\tilde{\vec{r}}} E_j(\omega,\vec{k}) = \sigma_{ij}(\omega,\vec{k}) E_j(\omega,\vec{k})$$

Tensor of complex dielectric permittivity (dielectric tensor)

$$\varepsilon_{ij}(\omega,\vec{k}) = \int_0^\infty d\tilde{t} \int d\tilde{\vec{r}}\, \hat{\varepsilon}_{ij}(\tilde{\vec{r}},\tilde{t}) e^{i\omega\tilde{t}-i\vec{k}\tilde{\vec{r}}}$$

Tensor of complex conductivity

$$\sigma_{ij}(\omega,\vec{k}) = \int_0^\infty d\tilde{t} \int d\tilde{\vec{r}}\, \hat{\sigma}_{ij}(\tilde{\vec{r}},\tilde{t}) e^{i\omega\tilde{t}-i\vec{k}\tilde{\vec{r}}}$$

$\varepsilon_{ij}(\omega,\vec{k}) \leftrightarrow \sigma_{ij}(\omega,\vec{k})$

$$\vec{D}(t,\vec{r}) = \varepsilon_0 \vec{E}(t,\vec{r}) + \int_{-\infty}^{t} dt' \vec{j}(t',\vec{r}) \quad (3.4)$$

$$D_i(\omega,\vec{k}) = \varepsilon_0 \varepsilon_{ij}(\omega,\vec{k}) E_j(\omega,\vec{k})$$
$$j_i(\omega,\vec{k}) = \sigma_{ij}(\omega,\vec{k}) E_j(\omega,\vec{k})$$

$$\vec{E}(t,\vec{r}) = \int_{-\infty}^{\infty} d\omega \int d\vec{\boldsymbol{k}}\ \vec{E}(\omega,\vec{k}) e^{-i\omega t + i\vec{k}\vec{r}}$$

$$\varepsilon_{ij}(\omega,\vec{k}) = \delta_{ij} + \frac{i}{\varepsilon_0 \omega} \sigma_{ij}(\omega,\vec{k})$$

1. unit tensor $\delta_{ij} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$
2. $\omega \neq 0$

$\varepsilon_{ij}(\omega,\vec{k})$ of the Isotropic Medium

M: Tensor Calculus

Rotational invariance of the second rank tensor $\varepsilon_{ij}(\omega,\vec{k})$

⇩

The second rank tensor $\varepsilon_{ij}(\omega,\vec{k})$ is a combination of δ_{ij} and $k_i k_j$

$$\varepsilon_{ij}(\omega,\vec{\boldsymbol{k}}) = \left(\delta_{ij} - \frac{k_i k_j}{k^2}\right)\varepsilon^{tr}(\omega,k) + \frac{k_i k_j}{k^2}\varepsilon^{l}(\omega,k)$$

Transverse dielectric permittivity

Longitudinal dielectric permittivity

$\sigma_{ij}(\omega, \vec{k})$ of the Isotropic Medium

The tensor of the complex conductivity $\sigma_{ij}(\omega, \vec{k})$:

$$\sigma_{ij}(\omega, \vec{\boldsymbol{k}}) = \left(\delta_{ij} - \frac{k_i k_j}{k^2}\right)\sigma^{tr}(\omega, k\,) + \frac{k_i k_j}{k^2}\sigma^{l}(\omega, k\,)$$

$$k = |\vec{\boldsymbol{k}}|$$

$$\varepsilon_{ij}(\omega, \vec{k}) = \delta_{ij} + \frac{i}{\varepsilon_0 \omega}\sigma_{ij}(\omega, \vec{k}) \Rightarrow \varepsilon^{tr,l}(\omega, k) = 1 + \frac{i}{\varepsilon_0 \omega}\sigma^{tr,l}(\omega, k)$$

Properties of tensors $\varepsilon_{ij}(\omega,\vec{k})$ and $\sigma_{ij}(\omega,\vec{k})$

$$\varepsilon_{ij}(\omega,\vec{k}) = \int_0^\infty d\tilde{t} \int d\vec{\tilde{r}}\; \hat{\varepsilon}_{ij}(\vec{\tilde{r}},\tilde{t})\, e^{i\omega\tilde{t}-i\vec{k}\vec{\tilde{r}}}$$

$$\sigma_{ij}(\omega,\vec{k}) = \int_0^\infty d\tilde{t} \int d\vec{\tilde{r}}\; \hat{\sigma}_{ij}(\vec{\tilde{r}},\tilde{t})\, e^{i\omega\tilde{t}-i\vec{k}\vec{\tilde{r}}}$$

complex

Real

$$\varepsilon_{ij}(\omega,\vec{k}) = \{\varepsilon_{ij}(-\omega,-\vec{k})\}^*$$

$$Re\left[\varepsilon_{ij}(\omega,\vec{k})\right] = Re\left[\{\varepsilon_{ij}(-\omega,-\vec{k})\}^*\right]$$

$$Im\left[\varepsilon_{ij}(\omega,\vec{k})\right] = -Im\left[\{\varepsilon_{ij}(-\omega,-\vec{k})\}^*\right]$$

"*" means complex conjugation

NB: The same true for:

1. Conductivity tensor $\sigma_{ij}(\omega,\vec{k})$
2. Isotropic medium → $\varepsilon^{tr,l}(\omega,k), \sigma^{tr,l}(\omega,k)$

More properties of material tensors

Mirror reflection of the coordinate axes
for the second rank (true=not pseudotensor) tensor $\varepsilon_{ij}(\omega, \vec{k})$

⇩

The second rank tensor $\varepsilon_{ij}(\omega, \vec{k})$ does not change the sign:

$$\varepsilon_{ij}(\omega, \vec{\boldsymbol{k}}) = \varepsilon_{ji}(\omega, -\vec{\boldsymbol{k}})$$

If + external magnetic field $\vec{\boldsymbol{B}}_0$ (pseudovector):

$$\varepsilon_{ij}(\omega, \vec{\boldsymbol{k}}, \vec{\boldsymbol{B}}_0) = \varepsilon_{ji}(\omega, -\vec{\boldsymbol{k}}, -\vec{\boldsymbol{B}}_0)$$

Electromagnetic waves in the medium

Dispertsion equation $\omega = \omega(\vec{k})$ for the medium (w/o external sources)

$$\nabla\vec{D} = 0$$
$$\nabla\vec{B} = 0$$
$$[\nabla \times \vec{E}] = -\frac{\partial \vec{B}}{\partial t}$$
$$[\nabla \times \vec{B}] = \frac{1}{c^2\varepsilon_0}\frac{\partial \vec{D}}{\partial t}$$

⇒

$$\propto \int_{-\infty}^{\infty} d\omega \int d\boldsymbol{\vec{k}}\ \vec{E}(\omega,\vec{k})e^{-i\omega t+i\vec{k}\vec{r}}$$

$$D_i(\omega,\vec{k}) = \varepsilon_{ij}(\omega,\vec{k})E_j(\omega,\vec{k})$$

⇒

$$k_i\varepsilon_{ij}(\omega,\vec{k})E_j = 0$$
$$k_iB_i = 0$$
$$[\vec{k} \times \vec{E}] = \omega\vec{B}$$
$$c^2[\vec{k} \times \vec{B}] = -\omega\varepsilon_{ij}(\omega,\vec{k})E_j$$

⇒

$$\vec{B} = \frac{1}{\omega}[\vec{k} \times \vec{E}]$$
$$[\vec{k} \times \vec{B}] = -\frac{\omega}{c^2}\varepsilon_{ij}(\omega,\vec{k})E_j$$

⇒

$$[\vec{k} \times [\vec{k} \times \vec{E}]] = -\frac{\omega^2}{c^2}\varepsilon_{ij}(\omega,\vec{k})E_j$$

⇓

$$\vec{k}(\vec{k}\cdot\vec{E}) - \vec{E}(\vec{k}\cdot\vec{k}) = -\frac{\omega^2}{c^2}\varepsilon_{ij}(\omega,\vec{k})E_j$$

⇓

$$\left\{k^2\delta_{ij} - k_ik_j - \frac{\omega^2}{c^2}\varepsilon_{ij}(\omega,\vec{k})\right\}E_j = 0$$

Electromagnetic waves in the medium

$$\left\{k^2\delta_{ij} - k_i k_j - \frac{\omega^2}{c^2}\varepsilon_{ij}(\omega,\vec{k})\right\}E_j = 0$$

Dispersion equation:

$$\Lambda = \left|k^2\delta_{ij} - k_i k_j - \frac{\omega^2}{c^2}\varepsilon_{ij}(\omega,\vec{k})\right| = 0$$

Electromagnetic waves in the isotropic medium

$$\left\{k^2\delta_{ij} - k_i k_j - \frac{\omega^2}{c^2}\varepsilon_{ij}(\omega,\vec{k})\right\}E_j = 0$$

$$\Lambda = \left|k^2\delta_{ij} - k_i k_j - \frac{\omega^2}{c^2}\varepsilon_{ij}(\omega,\vec{k})\right| = 0$$

$$\varepsilon_{ij}(\omega,\vec{\boldsymbol{k}}) = \left(\delta_{ij} - \frac{k_i k_j}{k^2}\right)\varepsilon^{tr}(\omega,k) + \frac{k_i k_j}{k^2}\varepsilon^{l}(\omega,k)$$

⇩

$$\left\{k^2\delta_{ij} - k_i k_j - \frac{\omega^2}{c^2}\left[\left(\delta_{ij} - \frac{k_i k_j}{k^2}\right)\varepsilon^{tr}(\omega,k) + \frac{k_i k_j}{k^2}\varepsilon^{l}(\omega,k)\right]\right\}E_j = 0$$

⇩

$$k^2\vec{E}^{tr} - \frac{\omega^2}{c^2}\left[\varepsilon^{tr}(\omega,k)\vec{E}^{tr} + \varepsilon^{l}(\omega,k)\vec{E}^{tr}\right] = 0$$

Longitudinal waves ⇨ $$\varepsilon^{l}(\omega,k) = 0$$

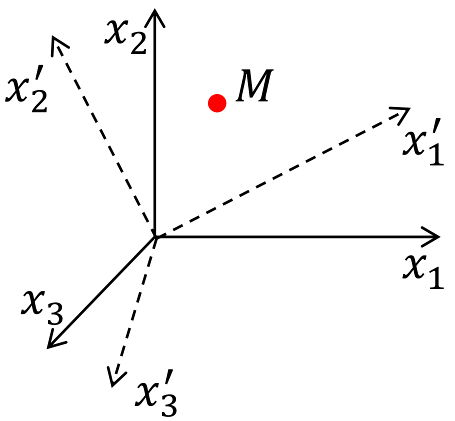
Transverse waves ⇨ $$\left[k^2 - \frac{\omega^2}{c^2}\varepsilon^{tr}(\omega,k)\right]^2 = 0$$

Summary

- Maxwell Equations
  - $\vec{D}$
  - Boundary conditions
- Complex tensors $\varepsilon_{ij}(\omega,\vec{k})$ and $\sigma_{ij}(\omega,\vec{k})$
  - Homogeneous medium
  - Isotropic medium
- Electromagnetic waves in medium
  - Isotropic medium: $\varepsilon^{l}(\omega,k) = 0;\ \left[k^2 - \frac{\omega^2}{c^2}\varepsilon^{tr}(\omega,k)\right]^2 = 0$
- Next:
  - For any classical equilibrium medium

    $\mu(0,k) = \mathbf{1}$ (Bohr-Van-Leeuwen theorem)
  - Energy of electromagnetic waves in the medium

M: Elements of Tensor Calculus



Coordinates of M: $x_k = e_{kj} x_j'$, $\quad x_k' = e_{kj}^{-1} x_j = e_{jk} x_j$

e_{ij} - cosine of the angles between axes

	x_1'	x_2'	x_3'
x_1	e_{11}	e_{12}	e_{13}
x_2	e_{21}	e_{22}	e_{23}
x_3	e_{31}	e_{32}	e_{33}

$$e_{ik} e_{jk} = e_{ki} e_{kj} = \delta_{ij} = \begin{cases} 1 & \text{for } i = j, \\ 0 & \text{for } i \neq j. \end{cases}$$

$$e_{ik} e_{jk} \rightarrow \sum_{k=1}^{3} e_{ik} e_{jk}$$

$$e_{ij} = \begin{pmatrix} e_{11} & e_{12} & e_{13} \\ e_{21} & e_{22} & e_{23} \\ e_{31} & e_{32} & e_{33} \end{pmatrix}$$

Transformation matrix of an orthogonal affine transformation

$$\boldsymbol{a} \cdot \boldsymbol{b} = a_i b_i = a_1 b_1 + a_2 b_2 + a_3 b_3 \qquad a_i b_i = a_i' b_i'$$

Invariant! → vector (tensor) definition

$\boldsymbol{x} = (x_1, x_2, x_3) \qquad a_i = (a_1, a_2, a_3) \qquad F_1 = a_i x_i$ =inv

A second-rank tensor $\boldsymbol{x} = (x_1, x_2, x_3)$, $\boldsymbol{y} = (y_1, y_2, y_3)$ $\qquad F_2 = d_{ij} x_i y_j$ =inv

M: Elements of Tensor Calculus

A second-rank tensor $\boldsymbol{x} = (x_1, x_2, x_3)$

$\boldsymbol{y} = (y_1, y_2, y_3)$

$F_2 = d_{ij} x_i y_j$ =inv

E.g. $\delta_{ij} x_i y_i = x_j y_j = \boldsymbol{x} \cdot \boldsymbol{y} = \text{const.}$

the transformation law of second-rank tensors

$$d_{ij} x_i x_j = d'_{ij} x'_i y'_i = d'_{ij} e_{mi} x_m e_{nj} y_n = d'_{sk} e_{is} e_{jk} x_i y_j$$

$$d_{ks} = e_{ki} e_{sj} d'_{ij}$$

Analogously we obtain $d'_{ks} = e_{ik} e_{js} d_{ij}$

Thus, a second-rank tensor is transformed like the outer product of two vectors $a_i b_j$. Therefore, one can define such a tensor as a set of nine quantities being transformed like the outer product of two vectors.

Tensors of higher rank are defined analogously. Thus, a third-rank tensor β_{ijk} is a set of 27 quantities, leaving the cubic form

$$F_3 = \beta_{ijk} x_i y_j z_k$$

M: Elements of Tensor Calculus

B.1.2 Compose the general second-rank tensor ε_{ij} with $\varepsilon_{ij}(\boldsymbol{k}) = \varepsilon_{ji}(-\boldsymbol{k})$ for real $\boldsymbol{k}$. For $\boldsymbol{k} \doteq 0$ reduce ε_{ij} over the indices, i.e., take the sum ε_{ii}.

Solution.

$$\varepsilon_{ij}(\boldsymbol{k}) = \alpha_1 \delta_{ij} + \alpha_2 k_i k_j = \left(\delta_{ij} - \frac{k_i k_j}{k^2}\right) \varepsilon^{\mathrm{tr}} + \frac{k_i k_j}{k^2} \varepsilon^{\mathrm{lo}} ,$$

i.e.,

$$\alpha_1 = \varepsilon^{\mathrm{tr}} , \qquad \alpha_2 = \frac{\varepsilon^{\mathrm{lo}} - \varepsilon^{\mathrm{tr}}}{k^2} .$$

In the limit $\boldsymbol{k} \to 0$ we have $\varepsilon_{ij}(0) = \alpha_1 \delta_{ij}$ and $\varepsilon^{\mathrm{lo}} = \varepsilon^{\mathrm{tr}} = \varepsilon$. For vanishing $\boldsymbol{k}$, $\varepsilon_{ij}(0) = \varepsilon \delta_{ij}$ is the general second-rank tensor. The reduction over the indices reads

$$\varepsilon_{ij}(\boldsymbol{k}) = (3-1)\,\varepsilon^{\mathrm{tr}} + \varepsilon^{\mathrm{lo}} = 2\,\varepsilon^{\mathrm{tr}} + \varepsilon^{\mathrm{lo}}, \qquad \varepsilon_{ii}^{\mathrm{lo}}(0) = 3\,\varepsilon .$$

M: Elements of Tensor Calculus

The components of a tensor can be both real and complex. Therefore, in general, we have to deal with complex tensors. Then, the concept of the Hermiticity of a tensor is important. A second-rank tensor is called Hermitian if ("*" means complex conjugation)

$$\alpha_{ij}^{*\mathrm{H}} = \alpha_{ji}^{\mathrm{H}}$$

but if $\alpha_{ij}^{*\mathrm{a}} = -\alpha_{ji}^{\mathrm{a}}$

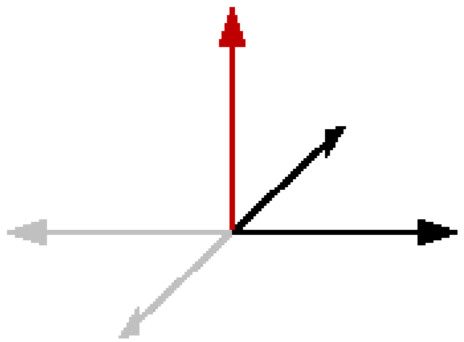
the tensor is called anti-Hermitian. Any tensor can be decomposed into a Hermitian and an anti-Hermitian part.

M: Elements of Tensor Calculus

True- and pseudo-tensors

…a pseudotensor is usually a quantity that transforms like a tensor under an orientation preserving coordinate transformation (e.g., a proper rotation), but ***additionally*** *changes sign under an* ***orientation reversing coordinate*** *transformation (e.g., an improper rotation, which is a transformation that can be expressed as a proper rotation followed by reflection). (Wikipedia)*

Pseudotensor by **orientation reversing coordinate** transformation:
- Even rank (0→scalar,2,..) ➔ change the sign (-1)
- Odd rank (1→vector,3,..) ➔ invariant (+1)



Under inversion the two vectors change sign, but their **cross product** is invariant [black are the two original vectors, grey are the inverted vectors, and red is their mutual cross product].

The completely antisymmetric third-rank unit tensor:

$$e_{ijk} \begin{cases} 0 & \text{if two of the indices } i, j, k \text{ coincide,} \\ 1 & \text{if the indices } i, j, k \text{ form a regular succession of the numbers } 1, 2, 3, \\ -1 & \text{if the indices } i, j, k \text{ form an irregular succession of the numbers } 1, 2, 3, \end{cases}$$

$$[\vec{a} \times \vec{b}]_i = e_{ijk} a_j b_k$$

M: Elements of Tensor Calculus

True- and pseudo-vectors and scalars:

[true vector × true vector] = pseudovector
[pseudovector × pseudovector] = pseudovector
[true vector × pseudovector] = true vector

(true vector · true vector) = true scalar
(true vector · pseudovector) = pseudo scalar
(pseudovector · pseudovector) = true scalar

!Also opeartors:

e.g.:

$$\left(\vec{\nabla}\cdot\ \Phi\right) = \frac{\partial\Phi}{\partial\vec{r}} = grad\ \Phi$$

$$\left[\vec{\nabla}\times\vec{a}\right] = curl\ \vec{a}$$

E.g.:

- True scalars: $t, \rho, \mathcal{E}(\vec{p}), \omega\left(\vec{k}\right)$
- True vectors$\vec{r}, \vec{p}, \vec{k}, \vec{D}, \vec{j}\ , \vec{v} = \frac{\partial\mathcal{E}(\vec{p})}{\partial\vec{p}}, \vec{v}_g = \frac{\partial\omega\left(\vec{k}\right)}{\partial\vec{k}}$
- Pseudovector: ???