Plasma dispersion function and Landau damping

M.Krasilnikov
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L#7

Discussed last times…

- Simplest Plasma Models: the Model of Independent Particles:
  - Including collisions ν_e
  - Spectra for waves in plasma + damping constants for high and low frequency cases (normal skin-effect $\lambda_{sk} \sim \frac{1}{\sqrt{\omega}}$)

Difficulties of the model:

- Static limit for || plasma oscillations ($\omega \to 0 \Rightarrow \varepsilon \to \infty;\ \Phi \to 0?$)
- Non-dissipative skin-effect $\sim c/\omega_p$) + anomalous skin-effect (($\lambda_{sk} \sim \omega^{-1/3}$))
- Can not explain low frequency longitudinal waves with linear dispersion (experimentally observed)

- Kinetic approach to plasma:
  - Vlasov equation $\boldsymbol{\delta f_\alpha(\omega, \vec{k}) = -i \frac{e_\alpha \vec{E} \cdot \frac{\partial f_{0\alpha}}{\partial \vec{p}}}{\omega - \vec{k}\vec{v}}}$
  - Dielectric permittivity from the kinetic model $\varepsilon_{ij}(\omega, \vec{k}) = \delta_{ij} + \sum_\alpha \frac{{e_\alpha}^2}{\varepsilon_0 \omega} \int \frac{v_i \left(\frac{\partial f_{0\alpha}}{\partial p_j}\right)}{\omega - \vec{k}\vec{v}} d\vec{p}$
  - Landau prescription
  - Cherenkov wave absorption
  - Longitudinal and transverse dielectric permittivity for the homogeneous isotropic plasma

Dielectric permittivity from the kinetic model

Homogeneous isotropic plasma:

$$\varepsilon^{l}(\omega,k) = 1 + \sum_{\alpha} \frac{{e_\alpha}^2}{\varepsilon_0 \omega k^2} \int d\vec{p}\, \frac{\left(\vec{k}\vec{v}\right)^2}{\omega - \vec{k}\vec{v}} \left(\frac{\partial f_{0\alpha}}{\partial \mathcal{E}_\alpha}\right)$$

$$\varepsilon^{tr}(\omega,k) = 1 + \sum_{\alpha} \frac{{e_\alpha}^2}{2\varepsilon_0 \omega k^2} \int d\vec{p}\, \frac{\left[\vec{k} \times \vec{v}\right]^2}{\omega - \vec{k}\vec{v}} \left(\frac{\partial f_{0\alpha}}{\partial \mathcal{E}_\alpha}\right)$$

- Maxwell $f_{0\alpha}(\mathcal{E}_\alpha) = f_{M\alpha} = \frac{N_\alpha}{(2\pi m_\alpha k T_\alpha)^{3/2}} exp\left(-\frac{\mathcal{E}_\alpha}{\kappa T_\alpha}\right)$
- Fermi distribution function $f_{0\alpha}(\mathcal{E}_\alpha) = f_{F\alpha} = \frac{\frac{2}{(2\pi\hbar)^{3/2}}}{exp\left(\frac{\mathcal{E}_\alpha - \mathcal{E}_{F\alpha}}{\kappa T_\alpha}\right)+1}$

$\mathcal{E}_\alpha = \frac{{p_\alpha}^2}{2m_\alpha}$

Longitudinal dielectric permittivity for homogeneous Maxwellian plasma

$$\varepsilon^l(\omega, k\ ) = 1 + \sum_\alpha \frac{{e_\alpha}^2}{\varepsilon_0 \omega k^2} \int d\vec{p}\ \frac{\left(\vec{k}\vec{v}\right)^2}{\omega - \vec{k}\vec{v}} \left(\frac{\partial f_{0\alpha}}{\partial \mathcal{E}_\alpha}\right)$$

- Maxwell $f_{0\alpha}(\mathcal{E}_\alpha) = f_{M\alpha} = \frac{N_\alpha}{(2\pi m_\alpha k T_\alpha)^{3/2}} exp\left(-\frac{\mathcal{E}_\alpha}{\kappa T_\alpha}\right)$

$$\frac{\partial f_{M\alpha}}{\partial \mathcal{E}_\alpha} = -\frac{1}{\kappa T_\alpha} f_{M\alpha}$$

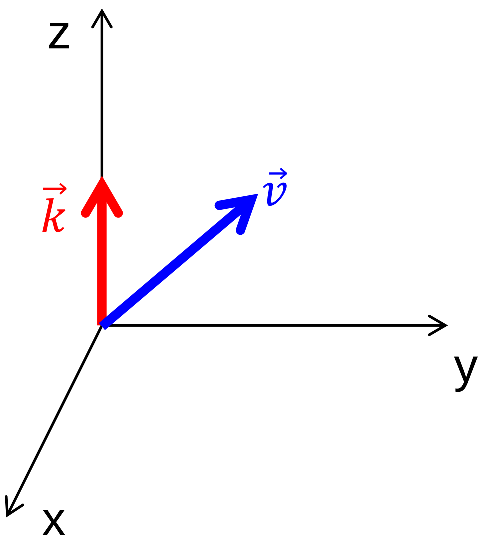
$$\varepsilon^l(\omega, k\ ) = 1 - \sum_\alpha \frac{{e_\alpha}^2}{\varepsilon_0 \omega k^2 \kappa T_\alpha} \int d\vec{p}\ \frac{\left(\vec{k}\vec{v}\right)^2}{\omega - \vec{k}\vec{v}} f_{M\alpha}$$

Longitudinal dielectric permittivity for homogeneous Maxwellian plasma

$$\varepsilon^l(\omega,k) = 1 - \sum_\alpha \frac{{e_\alpha}^2}{\varepsilon_0 \omega k^2 \kappa T_\alpha} \int d\vec{p}\, \frac{(\vec{k}\vec{v})^2}{\omega - \vec{k}\vec{v}} f_{M\alpha}$$

$$f_{M\alpha} = \frac{N_\alpha}{(2\pi m_\alpha k T_\alpha)^{3/2}} exp\left(-\frac{\mathcal{E}_\alpha}{\kappa T_\alpha}\right)$$

$$\mathcal{E}_\alpha = \frac{{p_\alpha}^2}{2m_\alpha} = \frac{m_\alpha {v_\alpha}^2}{2}$$



$$\int dp_x dp_y \rightarrow (2\pi m_\alpha k T_\alpha)^{2/2}$$

$$\varepsilon^l(\omega,k) = 1 - \sum_\alpha \frac{{e_\alpha}^2}{\varepsilon_0 \omega k^2 \kappa T_\alpha} \int_{-\infty}^{\infty} dp_z \frac{(kv_z)^2}{\omega - kv_z} \cdot \frac{N_\alpha}{(2\pi m_\alpha k T_\alpha)^{1/2}} exp\left(-\frac{{v_z}^2}{2{v_{T\alpha}}^2}\right)$$

$${v_{T\alpha}}^2 = \frac{kT_\alpha}{m_\alpha}$$

Longitudinal dielectric permittivity for homogeneous Maxwellian plasma

$$\varepsilon^l(\omega, k) = 1 - \sum_\alpha \frac{e_\alpha{}^2}{\varepsilon_0 \omega k^2 \kappa T_\alpha} \int_{-\infty}^{\infty} dp_z \frac{(kv_z)^2}{\omega - kv_z} \cdot \frac{N_\alpha}{(2\pi m_\alpha k T_\alpha)^{1/2}} exp\left(-\frac{v_z{}^2}{2v_{T\alpha}{}^2}\right)$$

$$kT_\alpha = m_\alpha v_{T\alpha}{}^2$$

$$\varepsilon^l(\omega, k) = 1 - \sum_\alpha \frac{N_\alpha e_\alpha{}^2}{\varepsilon_0 \omega k^2 m_\alpha v_{T\alpha}{}^2} \frac{1}{\left(2\pi m_\alpha m_\alpha v_{T\alpha}{}^2\right)^{1/2}} \int_{-\infty}^{\infty} dp_z \frac{(kv_z)^2}{\omega - kv_z} \cdot exp\left(-\frac{v_z{}^2}{2v_{T\alpha}{}^2}\right)$$

$$\varepsilon^l(\omega, k) = 1 - \sum_\alpha \frac{N_\alpha e_\alpha{}^2}{\varepsilon_0 m_\alpha} \frac{1}{\omega k^2 v_{T\alpha}{}^2} \frac{1}{\sqrt{2\pi} m_\alpha v_{T\alpha}} \int_{-\infty}^{\infty} dp_z \frac{(kv_z)^2}{\omega - kv_z} \cdot exp\left(-\frac{v_z{}^2}{2v_{T\alpha}{}^2}\right)$$

Longitudinal dielectric permittivity for homogeneous Maxwellian plasma

$$\varepsilon^{l}(\omega,k) = 1 - \sum_{\alpha}\frac{N_\alpha {e_\alpha}^2}{\varepsilon_0 m_\alpha}\frac{1}{\omega k^2 {v_{T\alpha}}^2}\frac{1}{\sqrt{2\pi}m_\alpha v_{T\alpha}}\int_{-\infty}^{\infty} dp_z \frac{(kv_z)^2}{\omega - kv_z}\cdot exp\left(-\frac{{v_z}^2}{2{v_{T\alpha}}^2}\right)$$

$$z = \frac{\omega}{\sqrt{2}kv_{T\alpha}}$$

$$p_z = m_\alpha v_z$$

$$t = \frac{v_z}{\sqrt{2}v_{T\alpha}}$$

$$\varepsilon^{l}(\omega,k) = 1 - \sum_{\alpha}\frac{{\omega_{p\alpha}}^2}{k^2{v_{T\alpha}}^2}\frac{1}{\sqrt{\pi}z}\int_{-\infty}^{\infty}\frac{t^2\cdot exp(-t^2)}{z-t}dt$$

$$-\sqrt{\pi}z - z^2\int_{-\infty}^{\infty}\frac{exp(-t^2)}{z-t}dt$$

$$\varepsilon^{l}(\omega,k) = 1 + \sum_{\alpha}\frac{{\omega_{p\alpha}}^2}{k^2{v_{T\alpha}}^2}\left\{1 + \frac{z}{\sqrt{\pi}}\int_{-\infty}^{\infty}\frac{exp(-t^2)}{z-t}dt\right\}$$

Plasma dispersion function

$$\varepsilon^{l}(\omega,k) = 1 + \sum_{\alpha} \frac{{\omega_{p\alpha}}^2}{k^2 {v_{T\alpha}}^2} \left\{ 1 + \frac{z}{\sqrt{\pi}} \int_{-\infty}^{\infty} \frac{exp(-t^2)}{z-t} dt \right\} = 1 + \sum_{\alpha} \frac{{\omega_{p\alpha}}^2}{k^2 {v_{T\alpha}}^2} \{1 - z \cdot \varsigma(z)\}$$

$$\varsigma(z) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} \frac{exp(-t^2)}{t-z} dt$$

$$z \cdot \varsigma(z) = \sqrt{2} z e^{-z^2} \int_{i\infty}^{\sqrt{2}z} e^{\frac{\tau^2}{2}} d\tau = -i\sqrt{2} z e^{-z^2} \{1 - \mathrm{erf}(iz\sqrt{2})\} = \Pi(z\sqrt{2})$$

$$z = \frac{\omega}{\sqrt{2} k v_{T\alpha}}$$

Longitudinal dielectric permittivity for homogeneous Maxwellian plasma

$$\varepsilon^{l}(\omega,k) = 1 + \sum_{\alpha} \frac{{\omega_{p\alpha}}^{2}}{k^{2}{v_{T\alpha}}^{2}} \left\{ 1 + \frac{z}{\sqrt{\pi}} \int_{-\infty}^{\infty} \frac{exp(-t^{2})}{z-t} dt \right\} = 1 + \sum_{\alpha} \frac{{\omega_{p\alpha}}^{2}}{k^{2}{v_{T\alpha}}^{2}} \left\{ 1 - \Pi\left(\frac{\omega}{k v_{T\alpha}}\right) \right\}$$

$$\Pi(z) = z e^{-\frac{z^{2}}{2}} \int_{i\infty}^{z} e^{\frac{\tau^{2}}{2}} d\tau = -iz e^{-\frac{z^{2}}{2}} \{1 - \mathrm{erf}(iz)\}$$

Useful asymptotic expansions:

A1) $|z| \ll 1 \quad \Rightarrow \quad \Pi(z) \approx -i\sqrt{\frac{\pi}{2}} z + O(z^{2})$

A2) $|z| \gg 1; Re(z) \gg Im(z) \Rightarrow \Pi(z) \approx 1 + \frac{1}{z^{2}} + \frac{3}{z^{4}} + O\left(\frac{1}{z^{5}}\right) - i\sqrt{\frac{\pi}{2}} z e^{-\frac{z^{2}}{2}} + O\left(\frac{1}{z^{6}}\right)$

A3) $|z| \gg 1; Re(z) \ll Im(z); Im(z) < 0 \Rightarrow \Pi(z) \approx -i\sqrt{2\pi} z e^{-\frac{z^{2}}{2}} + O\left(\frac{1}{z^{6}}\right)$

Transverse dielectric permittivity for homogeneous Maxwellian plasma

$$\varepsilon^{tr}(\omega,k) = 1 + \sum_{\alpha} \frac{{e_\alpha}^2}{2\varepsilon_0 \omega k^2} \int d\vec{p}\, \frac{\left[\vec{k}\times\vec{v}\right]^2}{\omega - \vec{k}\vec{v}} \left(\frac{\partial f_{0\alpha}}{\partial \mathcal{E}_\alpha}\right)$$

$$\varepsilon^{tr}(\omega,k) = 1 - \sum_{\alpha} \frac{{\omega_{p\alpha}}^2}{\omega^2} \cdot \Pi\left(\frac{\omega}{k v_{T\alpha}}\right)$$

$$\Pi(z) = z e^{-\frac{z^2}{2}} \int_{i\infty}^{z} e^{\frac{\tau^2}{2}} d\tau = -i z e^{-\frac{z^2}{2}} \{1 - \mathrm{erf}(iz)\}$$

Dielectric permittivity for homogeneous Maxwellian plasma

$$\varepsilon^{l}(\omega,k) = 1 + \sum_{\alpha} \frac{{\omega_{p\alpha}}^{2}}{k^{2}{v_{T\alpha}}^{2}} \left\{ 1 - \Pi\left(\frac{\omega}{k v_{T\alpha}}\right) \right\}$$

$$\varepsilon^{tr}(\omega,k) = 1 - \sum_{\alpha} \frac{{\omega_{p\alpha}}^{2}}{\omega^{2}} \cdot \Pi\left(\frac{\omega}{k v_{T\alpha}}\right)$$

$$\varepsilon^{l}(\omega,k) = 0$$

$$\left[k^{2} - \frac{\omega^{2}}{c^{2}} \varepsilon^{tr}(\omega,k)\right]^{2} = 0$$

$$\Pi(z) = z e^{-\frac{z^2}{2}} \int_{i\infty}^{z} e^{\frac{\tau^2}{2}} d\tau = -iz e^{-\frac{z^2}{2}} \{1 - \mathrm{erf}(iz)\}$$

A1) $|z| \ll 1 \quad \Rightarrow \quad \Pi(z) \approx -i\sqrt{\frac{\pi}{2}} z + O(z^2)$

A2) $|z| \gg 1; Re(z) \gg Im(z) \Rightarrow \Pi(z) \approx 1 + \frac{1}{z^2} + \frac{3}{z^4} + O\left(\frac{1}{z^5}\right) - i\sqrt{\frac{\pi}{2}} z e^{-\frac{z^2}{2}} + O\left(\frac{1}{z^6}\right)$

A3) $|z| \gg 1; Re(z) \ll Im(z); Im(z) < 0 \Rightarrow \Pi(z) \approx -i\sqrt{2\pi} z e^{-\frac{z^2}{2}} + O\left(\frac{1}{z^6}\right)$

Longitudinal Oscillations in static Limit ω→ 0

$$\varepsilon^l(\omega,k) = 1 + \sum_\alpha \frac{{\omega_{p\alpha}}^2}{k^2 {v_{T\alpha}}^2}\left\{1 - \Pi\left(\frac{\omega}{k v_{T\alpha}}\right)\right\}$$

$$\varepsilon^l(\omega,k) = 0$$

A1) $|z| \ll 1 \quad \Rightarrow \quad \Pi(z) \approx -i\sqrt{\frac{\pi}{2}}\,z + O(z^2)$

$$\varepsilon^l(\omega \to 0,k) = 1 + \sum_\alpha \frac{{\omega_{p\alpha}}^2}{k^2 {v_{T\alpha}}^2} = 1 + \frac{1}{k^2 {r_D}^2}$$

Problem 4.1 (Lecture 4): Electrostatic field of a point charge

$$\Phi(\vec r) = \frac{q}{\varepsilon_0 (2\pi)^3}\int d\vec k\, \frac{e^{i\vec k(\vec r - \vec r_0)}}{k_i \cdot k_j \cdot \varepsilon_{ij}(0,\vec k)}$$

• If $k_i k_j \varepsilon_{ij}(0,\vec k) = k^2 + \frac{1}{r_{scr}^2}$

$$\Phi(\vec r) = \frac{q}{\varepsilon_0 \cdot (2\pi)^3}\int d\vec k\, \frac{e^{i\vec k(\vec r - \vec r_0)}}{k^2 + \frac{1}{r_{scr}^2}} = \left[\begin{array}{l}\text{Problem 2,}\\ \text{lecture 2}\end{array}\right] = \frac{q}{4\pi\varepsilon_0 \cdot |\vec r - \vec r_0|}\, e^{-\frac{|\vec r - \vec r_0|}{r_{scr}}}$$

$$\frac{1}{{r_D}^2} = \sum_\alpha \frac{{e_\alpha}^2 N_\alpha}{\varepsilon_0 \kappa T_\alpha} = \sum_\alpha \frac{{\omega_{p\alpha}}^2}{{v_{T\alpha}}^2}$$

Lecture 2:
Debye length r_D-3

Transverse waves in static Limit $\omega \to 0$

$$\varepsilon^{tr}(\omega, k) = 1 - \sum_\alpha \frac{{\omega_{p\alpha}}^2}{\omega^2} \cdot \Pi\left(\frac{\omega}{k v_{T\alpha}}\right)$$

$$\left[k^2 - \frac{\omega^2}{c^2}\varepsilon^{tr}(\omega, k)\right]^2 = 0$$

A1) $|z| \ll 1 \quad \Rightarrow \quad \Pi(z) \approx -i\sqrt{\frac{\pi}{2}}z + O(z^2)$

$\varepsilon = 1 + \frac{i}{\varepsilon_0 \omega}\sigma$ ←(L6)→ $\lambda_{sk} = c\sqrt{\frac{2\varepsilon_0}{\omega\sigma}}$

$$\varepsilon^{tr}(\omega \to 0, k) = 1 - \sum_\alpha \frac{{\omega_{p\alpha}}^2}{\omega^2} \cdot \Pi\left(\frac{\omega}{k v_{T\alpha}}\right) \approx 1 + i\sqrt{\frac{\pi}{2}}\sum_\alpha \frac{{\omega_{p\alpha}}^2}{\omega k v_{T\alpha}} = 1 + \frac{i\sigma^{tr}(\omega \to 0, k)}{\varepsilon_0 \omega}$$

$$k^2 = \frac{\omega^2}{c^2}\varepsilon^{tr}(\omega \to, k) = \frac{\omega^2}{c^2} \cdot \left(1 + i\sqrt{\frac{\pi}{2}}\sum_\alpha \frac{{\omega_{p\alpha}}^2}{\omega k v_{T\alpha}}\right) \approx i\sqrt{\frac{\pi}{2}} \cdot \frac{\omega}{kc^2}\sum_\alpha \frac{{\omega_{p\alpha}}^2}{k v_{T\alpha}}$$

$$Im(k) \approx \left(\sqrt{\frac{\pi}{2}} \cdot \frac{\omega}{c^2}\sum_\alpha \frac{{\omega_{p\alpha}}^2}{v_{T\alpha}}\right)^{1/3}$$

$$\lambda_{sk} = \frac{1}{Im\, k} \approx \left(\sqrt{\frac{2}{\pi}} \cdot \frac{c^2 v_{Te}}{{\omega_{pe}}^2 \omega}\right)^{1/3} \sim \frac{1}{\omega^{1/3}}$$

Anomalous skin-effect!

High frequency longitudinal waves

$$\varepsilon^l(\omega,k) = 1 + \sum_\alpha \frac{{\omega_{p\alpha}}^2}{k^2{v_{T\alpha}}^2}\left\{1 - \Pi\left(\frac{\omega}{kv_{T\alpha}}\right)\right\}$$

$$\varepsilon^l(\omega,k) = 0$$

$$\omega \gg kv_{Te}, kv_{Ti}$$

A2) $|z| \gg 1; Re(z) \gg Im(z) \Rightarrow \Pi(z) \approx 1 + \frac{1}{z^2} + \frac{3}{z^4} + O\left(\frac{1}{z^5}\right) - i\sqrt{\frac{\pi}{2}}ze^{-\frac{z^2}{2}} + O\left(\frac{1}{z^6}\right)$

$$\varepsilon^l(\omega,k) = 1 + \sum_\alpha \frac{{\omega_{p\alpha}}^2}{k^2{v_{T\alpha}}^2}\left\{1 - \Pi\left(\frac{\omega}{kv_{T\alpha}}\right)\right\} \approx$$

$$\approx 1 + \sum_\alpha \frac{{\omega_{p\alpha}}^2}{k^2{v_{T\alpha}}^2}\left\{1 - 1 - \frac{1}{z^2} - \frac{3}{z^4} + i\sqrt{\frac{\pi}{2}}ze^{-\frac{z^2}{2}}\right\}\Bigg|_{z=\frac{\omega}{kv_{T\alpha}}}$$

High frequency longitudinal waves

$$\varepsilon^l(\omega, k) \approx 1 - \frac{{\omega_{pe}}^2}{k^2 {v_{Te}}^2}\left\{\frac{1}{z^2} + \frac{3}{z^4} - i\sqrt{\frac{\pi}{2}}\, z e^{-\frac{z^2}{2}}\right\}\Bigg|_{z=\frac{\omega}{k v_{T\alpha}}}$$

$$\varepsilon^l(\omega, k) \approx 1 - \frac{{\omega_{pe}}^2}{\omega^2}\left(1 + \frac{3k^2 {v_{Te}}^2}{\omega^2}\right) + i\sqrt{\frac{\pi}{2}}\frac{\omega {\omega_{pe}}^2}{k^3 {v_{Te}}^3} e^{-\frac{\omega^2}{2k^2 {v_{Te}}^2}}$$

Initial value problem (Lecture 5)

$$\Lambda(\omega,\vec{k}) = \left|k^2\delta_{ij} - k_i k_j - \frac{\omega^2}{c^2}\varepsilon_{ij}(\omega,\vec{k})\right| = 0$$

Assumptions:

- $Re\Lambda(\omega,\vec{k}) \gg Im\Lambda(\omega,\vec{k})$
- $\omega \to \omega + i\delta$

$$Re\Lambda(\omega,\vec{k}) + i\delta\frac{\partial Re\Lambda(\omega,\vec{k})}{\partial\omega} + i\,Im\Lambda(\omega,\vec{k}) - \delta\frac{\partial Im\Lambda(\omega,\vec{k})}{\partial\omega} = 0$$

=0

$$\boldsymbol{\delta = -\frac{Im\Lambda(\omega,\vec{k})}{\frac{\partial Re\Lambda(\omega,\vec{k})}{\partial\omega}}}$$

≤0 → damping decrement

>0 → instability increment (non-equilibrium medium)

Homogeneous isotropic medium: $\varepsilon_{ij}(\omega,\vec{\boldsymbol{k}}) = \left(\delta_{ij} - \frac{k_i k_j}{k^2}\right)\varepsilon^{tr}(\omega,k) + \frac{k_i k_j}{k^2}\varepsilon^{l}(\omega,k)$

Longitudinal waves ⇒ $Re\,\varepsilon^l(\omega,k) = 0; \quad \delta^l(k) = -\frac{Im\,\varepsilon^l(\omega,k)}{\frac{\partial Re\,\varepsilon^l(\omega,k)}{\partial\omega}}$

Transverse waves ⇒ $k^2 = \frac{\omega^2}{c^2}Re\,\varepsilon^{tr}(\omega,k); \quad \delta^{tr}(k) = -\frac{Im\,\omega^2\varepsilon^{tr}(\omega,k)}{\frac{\partial}{\partial\omega}\omega^2 Re\,\varepsilon^l(\omega,k)}$

$\frac{\partial Re\,\varepsilon^l(\omega,k)}{\partial\omega} > 0$
$\frac{\partial}{\partial\omega}\omega^2 Re\,\varepsilon^l(\omega,k) > 0$
For equil. medium

High frequency longitudinal waves

$$\varepsilon^l(\omega,k) \approx 1-\frac{{\omega_{pe}}^2}{\omega^2}\left(1+\frac{3k^2{v_{Te}}^2}{\omega^2}\right)+i\sqrt{\frac{\pi}{2}}\frac{\omega{\omega_{pe}}^2}{k^3{v_{Te}}^3}e^{-\frac{\omega^2}{2k^2{v_{Te}}^2}}$$

$$\boldsymbol{Re\,\varepsilon^l(\omega,k)=0}$$

$$Re\,\varepsilon^l(\omega,k) \approx 1-\frac{{\omega_{pe}}^2}{\omega^2}\left(1+\frac{3k^2{v_{Te}}^2}{\omega^2}\right)=0$$

$$\omega^2={\omega_{pe}}^2\left(1+\frac{3k^2{v_{Te}}^2}{\omega^2}\right)\approx{\omega_{pe}}^2\left(1+\frac{3k^2{v_{Te}}^2}{{\omega_{pe}}^2}\right)$$

$$\omega^2={\omega_{pe}}^2\left(1+3k^2{r_{De}}^2\right)$$

High frequency longitudinal waves

$$\varepsilon^l(\omega,k) \approx 1 - \frac{\omega_{pe}{}^2}{\omega^2}\left(1 + \frac{3k^2 v_{Te}{}^2}{\omega^2}\right) + i\sqrt{\frac{\pi}{2}}\frac{\omega\omega_{pe}{}^2}{k^3 v_{Te}{}^3} e^{-\frac{\omega^2}{2k^2 v_{Te}{}^2}}$$

$$Re\,\varepsilon^l(\omega,k) \approx 1 - \frac{\omega_{pe}{}^2}{\omega^2}\left(1 + \frac{3k^2 v_{Te}{}^2}{\omega^2}\right)$$

$$Im\,\varepsilon^l(\omega,k) \approx \sqrt{\frac{\pi}{2}}\frac{\omega\omega_{pe}{}^2}{k^3 v_{Te}{}^3} e^{-\frac{\omega^2}{2k^2 v_{Te}{}^2}}$$

$$\boldsymbol{\delta^l(k) = -\frac{Im\,\varepsilon^l(\omega,k)}{\frac{\partial Re\,\varepsilon^l(\omega,k)}{\partial\omega}}}$$

⇨

$$\delta^l(k) = -\sqrt{\frac{\pi}{8}}\frac{\omega^4}{k^3 v_{Te}{}^3} e^{-\frac{\omega^2}{2k^2 v_{Te}{}^2}}$$

$$\omega^2 = \omega_{pe}{}^2\left(1 + 3k^2 r_{De}{}^2\right)$$

$$\delta^l(k) = -\sqrt{\frac{\pi}{8}}\frac{\omega_{pe}{}^4\left(1 + 3k^2 r_{De}{}^2\right)^2}{k^3 v_{Te}{}^3} e^{-\frac{\omega_{pe}{}^2\left(1+3k^2 r_{De}{}^2\right)}{2k^2 v_{Te}{}^2}}$$

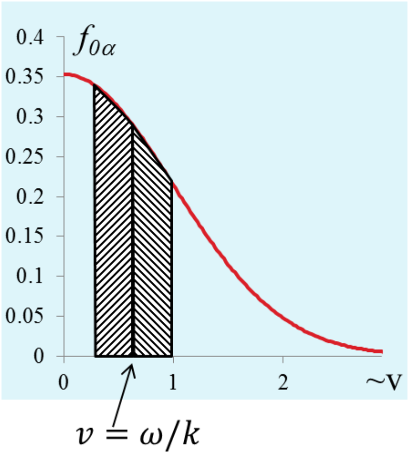
$$\delta^l(k) = -\sqrt{\frac{\pi}{8}}\cdot\frac{\omega_{pe}}{k^3 r_{De}{}^3}\cdot e^{-\frac{1}{2k^2 r_{De}{}^2}-\frac{3}{2}}$$

High frequency longitudinal waves

$$\omega^2 = {\omega_{pe}}^2\left(1 + 3k^2{r_{De}}^2\right)$$

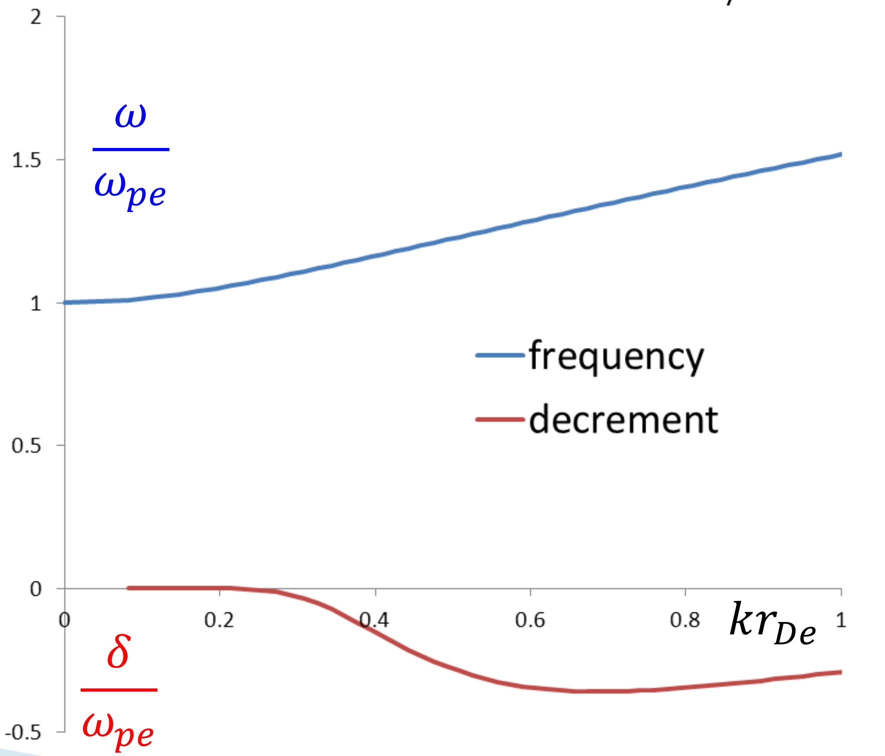
$$\delta^l(k) = -\sqrt{\frac{\pi}{8}} \cdot \frac{\omega_{pe}}{k^3{r_{De}}^3} \cdot e^{-\frac{1}{2k^2{r_{De}}^2}-\frac{3}{2}}$$

$\omega \gg kv_{Te}, kv_{Ti}$



Landau Damping:
For a lossless (still no collisions!) system there exist a physical solution for the oscillations characterized by an exponential decay corresponding to a damping

$kr_{De} \ll 1 \rightarrow \exp$ *small damping*



Summary

- Dielectric permittivity for homogeneous Maxwellian plasma:
  - Plasma dispersion function $\varsigma(z) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} \frac{exp(-t^2)}{t-z} dt$
  - Function $\Pi(z\sqrt{2}) = z \cdot \varsigma(z)$ and its asymptotic
  - Longitudinal dielectric permittivity $\varepsilon^l(\omega, k\ ) = 1 + \sum_\alpha \frac{{\omega_{p\alpha}}^2}{k^2 {v_{T\alpha}}^2} \left\{1 - \Pi\left(\frac{\omega}{k v_{T\alpha}}\right)\right\}$
  - Transverse dielectric permittivity $\varepsilon^{tr}(\omega, k\ ) = 1 - \sum_\alpha \frac{{\omega_{p\alpha}}^2}{\omega^2} \cdot \Pi\left(\frac{\omega}{k v_{T\alpha}}\right)$
- Longitudinal plasma oscillations in static limit → Debye shielding
- Transverse waves in static limit → anomalous skin-effect
- High frequency longitudinal waves → Landau damping
  - Next:
    - Longitudinal waves: intermediate frequency rang → ion-acoustic waves in nonisothermal plasma
    - Transverse high frequency waves in plasma
    - Degenerate plasma (Fermi distribution)
    - Oscillation and waves in degenerate collisionless plasma:
      - High frequency plasma waves and zero-point sound

Plasma Dispersion Function (derivation)

$$\zeta(z) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} \frac{e^{-t^2}}{t-z}\, dt$$

$$\zeta'(z) = \frac{d\zeta}{dz} = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} \frac{e^{-t^2}}{(t-z)^2}\, dt =$$

$$= -\frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-t^2} d\left(\frac{1}{t-z}\right) = -\frac{1}{\sqrt{\pi}} \left\{ \underbrace{\frac{e^{-t^2}}{t-z}\Bigg|_{-\infty}^{\infty}}_{0} + \int_{-\infty}^{\infty} \frac{2t e^{-t^2}}{t-z} \right\} =$$

$$= -\frac{2}{\sqrt{\pi}} \int_{-\infty}^{\infty} \frac{t-z+z}{t-z}\, e^{-t^2} dt =$$

$$= -\frac{2}{\sqrt{\pi}} \left\{ \underbrace{\int_{-\infty}^{\infty} e^{-t^2} dt}_{\to \sqrt{\pi}} + z \int_{-\infty}^{\infty} \frac{e^{-t^2}}{t-z}\, dt \right\} =$$

$$= -2 \left\{ 1 + z \cdot \zeta(z) \right\}$$

Plasma Dispersion Function (derivation)

$$\frac{d\mathcal{Z}}{dz} + 2z \cdot \mathcal{Z} = -2$$

$$\mathcal{Z}(0) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} \frac{e^{-t^2}}{t} dt = \frac{1}{\sqrt{\pi}} \left\{ v.P \int_{-\infty}^{\infty} \frac{e^{-t^2}}{t} dt + i\pi \cdot \delta(t) \right\}$$

$$\mathcal{Z}(0) = i\sqrt{\pi}$$

$$\mathcal{Z}(z) = e^{-z^2} \left(i\sqrt{\pi} - 2\int_0^z e^{x^2} dx \right)$$

Plasma Dispersion Function (derivation)

$$\zeta(z) = e^{-z^2}\left(i\sqrt{\pi} - 2\int_0^z e^{x^2}\,dx\right)$$

$$t = ix \qquad \int_{-\infty}^{0} e^{-t^2}\,dt = \frac{\sqrt{\pi}}{2}$$

$$\zeta(z) = 2i\,e^{-z^2}\int_{-\infty}^{iz} e^{-t^2}\,dt$$

$$-t^2 = \frac{\tau^2}{2} \qquad t = i\frac{\tau}{\sqrt{2}} \qquad dt = \frac{i}{\sqrt{2}}\,d\tau$$

$$t\Big|_{-\infty}^{iz} \longrightarrow \tau\Big|_{i\infty}^{\sqrt{2}z} \qquad \tau = -i\sqrt{2}\,t$$

Plasma Dispersion Function (derivation)

$$\zeta(z) = 2i e^{-z^2} \int_{-\infty}^{iz} e^{-t^2} dt$$

$$-t^2 = \frac{\tau^2}{2} \qquad t = i\frac{\tau}{\sqrt{2}} \qquad dt = \frac{i}{\sqrt{2}} d\tau$$

$$t\Big|_{-\infty}^{iz} \longrightarrow \tau\Big|_{i\infty}^{\sqrt{2}z} \qquad \tau = -i\sqrt{2}\,t$$

$$\zeta(z) = \sqrt{2}\, e^{-z^2} \int_{i\infty}^{\sqrt{2}z} e^{\tau^2/2}\, d\tau$$

$$z \cdot \zeta(z) = z\sqrt{2} \cdot e^{-\frac{(z\sqrt{2})^2}{2}} \int_{i\infty}^{\sqrt{2}z} e^{\tau^2/2} d\tau$$