Wave and oscillations in the collisionless degenerate plasma. Zero-point sound.

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L#8

Discussed last time…

- Dielectric permittivity for homogeneous Maxwellian plasma:
  - Plasma dispersion function $\varsigma(z) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} \frac{exp(-t^2)}{t-z} dt$
  - Function $\Pi(z\sqrt{2}) = z \cdot \varsigma(z)$ and its asymptotic
  - Longitudinal dielectric permittivity $\varepsilon^{l}(\omega, k) = 1 + \sum_{\alpha} \frac{{\omega_{p\alpha}}^2}{k^2 {v_{T\alpha}}^2} \left\{1 - \Pi\left(\frac{\omega}{k v_{T\alpha}}\right)\right\}$
  - Transverse dielectric permittivity $\varepsilon^{tr}(\omega, k) = 1 - \sum_{\alpha} \frac{{\omega_{p\alpha}}^2}{\omega^2} \cdot \Pi\left(\frac{\omega}{k v_{T\alpha}}\right)$
- Longitudinal plasma oscillations in static limit → Debye shielding
- Transverse waves in static limit → anomalous skin-effect
- High frequency longitudinal waves → Landau damping
  - Not discussed:
    - Longitudinal waves: intermediate frequency range → ion-acoustic waves in nonisothermal plasma
    - Transverse high frequency waves in plasma
    - Degenerate plasma (Fermi distribution)
    - Oscillation and waves in degenerate collisionless plasma:
      - High frequency plasma waves and zero-point sound

Dielectric permittivity for homogeneous Maxwellian plasma

$$\varepsilon^{l}(\omega,k) = 1 + \sum_{\alpha} \frac{{\omega_{p\alpha}}^{2}}{k^{2}{v_{T\alpha}}^{2}} \left\{1 - \Pi\left(\frac{\omega}{k v_{T\alpha}}\right)\right\}$$

$$\varepsilon^{tr}(\omega,k) = 1 - \sum_{\alpha} \frac{{\omega_{p\alpha}}^{2}}{\omega^{2}} \cdot \Pi\left(\frac{\omega}{k v_{T\alpha}}\right)$$

$$\varepsilon^{l}(\omega,k) = 0$$

$$\left[k^{2} - \frac{\omega^{2}}{c^{2}}\varepsilon^{tr}(\omega,k)\right]^{2} = 0$$

$$\Pi(z) = z e^{-\frac{z^2}{2}} \int_{i\infty}^{z} e^{\frac{\tau^2}{2}} d\tau = -iz e^{-\frac{z^2}{2}} \{1 - \mathrm{erf}(iz)\}$$

A1) $|z| \ll 1 \quad \Rightarrow \quad \Pi(z) \approx -i\sqrt{\frac{\pi}{2}}z + O(z^2)$

A2) $|z| \gg 1; Re(z) \gg Im(z) \Rightarrow \Pi(z) \approx 1 + \frac{1}{z^2} + \frac{3}{z^4} + O\left(\frac{1}{z^5}\right) - i\sqrt{\frac{\pi}{2}} z e^{-\frac{z^2}{2}} + O\left(\frac{1}{z^6}\right)$

A3) $|z| \gg 1; Re(z) \ll Im(z); Im(z) < 0 \Rightarrow \Pi(z) \approx -i\sqrt{2\pi} z e^{-\frac{z^2}{2}} + O\left(\frac{1}{z^6}\right)$

Transverse waves, high frequencies

$$\varepsilon^{tr}(\omega,k) = 1 - \sum_\alpha \frac{{\omega_{p\alpha}}^2}{\omega^2}\cdot\Pi\left(\frac{\omega}{k v_{T\alpha}}\right)$$

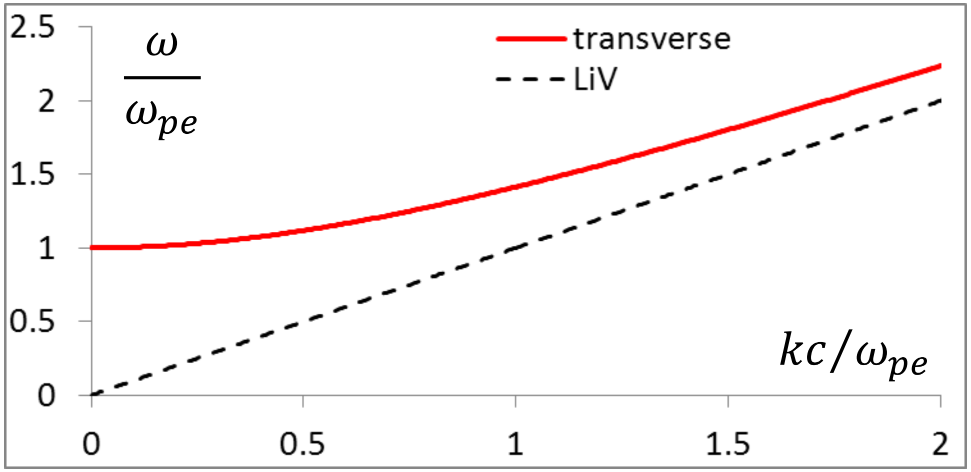
$$\left[k^2 - \frac{\omega^2}{c^2}\varepsilon^{tr}(\omega,k)\right]^2 = 0$$

A2) $|z| \gg 1; Re(z) \gg Im(z) \Rightarrow \Pi(z) \approx 1 + \frac{1}{z^2} + \frac{3}{z^4} + O\left(\frac{1}{z^5}\right) - i\sqrt{\frac{\pi}{2}}\, z e^{-\frac{z^2}{2}} + O\left(\frac{1}{z^6}\right)$

$$\varepsilon^{tr}(\omega,k) = 1 - \sum_\alpha \frac{{\omega_{p\alpha}}^2}{\omega^2}\cdot\Pi\left(\frac{\omega}{k v_{T\alpha}}\right) \approx 1 - \sum_\alpha \frac{{\omega_{p\alpha}}^2}{\omega^2}\cdot(1+\cdots)$$

$$k^2 = \frac{\omega^2}{c^2}\varepsilon^{tr}(\omega,k) = \frac{\omega^2 - {\omega_{pe}}^2}{c^2}$$

$$\omega^2 = {\omega_{pe}}^2 + k^2c^2$$



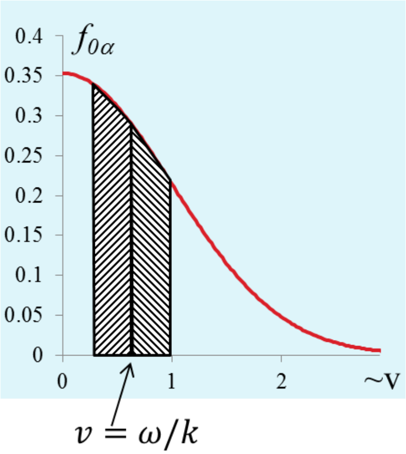
High frequency longitudinal waves

$$\omega^2 = {\omega_{pe}}^2\left(1 + 3k^2{r_{De}}^2\right)$$

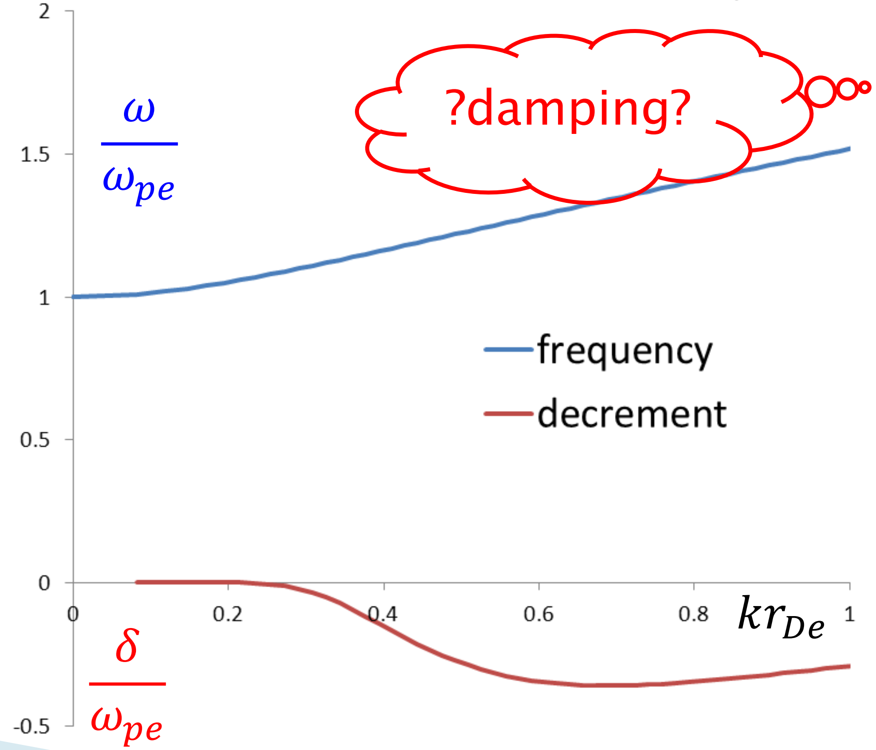
$$\delta^l(k) = -\sqrt{\frac{\pi}{8}} \cdot \frac{\omega_{pe}}{k^3{r_{De}}^3} \cdot e^{-\frac{1}{2k^2{r_{De}}^2}-\frac{3}{2}}$$

$\omega \gg kv_{Te}, kv_{Ti}$

A1) $|z| \ll 1 \;\Rightarrow\; \Pi(z) \approx -i\sqrt{\frac{\pi}{2}}z + O(z^2)$



Landau Damping:
For a lossless (still no collisions!) system there exist a physical solution for the oscillations characterized by an exponential decay corresponding to a damping



$kr_{De} \ll 1 \;\rightarrow\; \exp\;\; small\ damping$

Longitudinal oscillations, short wavelengths

$$\varepsilon^l(\omega, k) = 1 + \sum_\alpha \frac{{\omega_{p\alpha}}^2}{k^2 {v_{T\alpha}}^2}\left\{1 - \Pi\left(\frac{\omega}{k v_{T\alpha}}\right)\right\}$$

$$\varepsilon^l(\omega, k) = 0$$

A3) $|z| \gg 1; Re(z) \ll Im(z); Im(z) < 0 \Rightarrow \Pi(z) \approx -i\sqrt{2\pi} z e^{-\frac{z^2}{2}} + O\left(\frac{1}{z^6}\right)$

$\boldsymbol{k r_{De} \gg 1}$

$\omega \gg k v_{Te},\ k v_{Ti}$

$$\frac{\omega}{k v_{Te}} = \frac{\pi}{\sqrt{2\ln\left(k^2 {r_{De}}^2\right)}} - i\sqrt{2\ln\left(k^2 {r_{De}}^2\right)}$$

Longitudinal waves: intermediate frequency range

$$\varepsilon^l(\omega,k) = 1 + \sum_\alpha \frac{{\omega_{p\alpha}}^2}{k^2 {v_{T\alpha}}^2}\left\{1 - \Pi\left(\frac{\omega}{k v_{T\alpha}}\right)\right\}$$

$$\varepsilon^l(\omega,k) = 0$$

$$v_{Ti} \ll \frac{\omega}{k} \ll v_{Te}$$

A1) $|z| \ll 1 \quad \Rightarrow \quad \Pi(z) \approx -i\sqrt{\frac{\pi}{2}}z + O(z^2)$

A2) $|z| \gg 1; Re(z) \gg Im(z) \Rightarrow \Pi(z) \approx 1 + \frac{1}{z^2} + \frac{3}{z^4} + O\left(\frac{1}{z^5}\right) - i\sqrt{\frac{\pi}{2}} z e^{-\frac{z^2}{2}} + O\left(\frac{1}{z^6}\right)$

$$\varepsilon^l(\omega,k) = 1 + \frac{{\omega_{pe}}^2}{k^2 {v_{Te}}^2}\left\{1 - \Pi\left(\frac{\omega}{k v_{Te}}\right)\right\} + \frac{{\omega_{pi}}^2}{k^2 {v_{Ti}}^2}\left\{1 - \Pi\left(\frac{\omega}{k v_{Ti}}\right)\right\}$$

A1 A2

Longitudinal waves: intermediate frequency range

$$v_{Ti} \ll \frac{\omega}{k} \ll v_{Te}$$

A1) $|z| \ll 1 \quad \Rightarrow \quad \Pi(z) \approx -i\sqrt{\frac{\pi}{2}}z + O(z^2)$

A2) $|z| \gg 1; Re(z) \gg Im(z) \Rightarrow \Pi(z) \approx 1 + \frac{1}{z^2} + \frac{3}{z^4} + O\left(\frac{1}{z^5}\right) - i\sqrt{\frac{\pi}{2}}ze^{-\frac{z^2}{2}} + O\left(\frac{1}{z^6}\right)$

$$\varepsilon^l(\omega, k\,) = 1 + \frac{{\omega_{pe}}^2}{k^2{v_{Te}}^2}\left\{1 - \Pi\left(\frac{\omega}{kv_{Te}}\right)\right\} + \frac{{\omega_{pi}}^2}{k^2{v_{Ti}}^2}\left\{1 - \Pi\left(\frac{\omega}{kv_{Ti}}\right)\right\} \approx$$

$$\approx 1 + \frac{{\omega_{pe}}^2}{k^2{v_{Te}}^2}\left\{1 + i\sqrt{\frac{\pi}{2}}\frac{\omega}{kv_{Te}}\right\} + \frac{{\omega_{pi}}^2}{k^2{v_{Ti}}^2}\left\{1 - 1 - \frac{1}{z^2} - \frac{3}{z^4} + i\sqrt{\frac{\pi}{2}}ze^{-\frac{z^2}{2}}\right\}\Bigg|_{z=\frac{\omega}{kv_{Ti}}}$$

Longitudinal waves: intermediate frequency range

$$v_{Ti} \ll \frac{\omega}{k} \ll v_{Te}$$

$$\varepsilon^l(\omega,k) = 1 + \frac{{\omega_{pe}}^2}{k^2{v_{Te}}^2}\left\{1 - \Pi\left(\frac{\omega}{kv_{Te}}\right)\right\} + \frac{{\omega_{pi}}^2}{k^2{v_{Ti}}^2}\left\{1 - \Pi\left(\frac{\omega}{kv_{Ti}}\right)\right\} \approx$$

$$\approx 1 + \frac{{\omega_{pe}}^2}{k^2{v_{Te}}^2}\left\{1 + i\sqrt{\frac{\pi}{2}}\frac{\omega}{kv_{Te}}\right\} - \frac{{\omega_{pi}}^2}{k^2{v_{Ti}}^2}\left\{\frac{1}{z^2} + \frac{3}{z^4} - i\sqrt{\frac{\pi}{2}}ze^{-\frac{z^2}{2}}\right\}\Bigg|_{z=\frac{\omega}{kv_{Ti}}} =$$

$$\approx 1 + \frac{{\omega_{pe}}^2}{k^2{v_{Te}}^2}\left\{1 + i\sqrt{\frac{\pi}{2}}\frac{\omega}{kv_{Te}}\right\} - \frac{{\omega_{pi}}^2}{\omega^2}\left\{1 + 3\frac{k^2{v_{Ti}}^2}{\omega^2}\right\} + i\sqrt{\frac{\pi}{2}}\frac{\omega{\omega_{pi}}^2}{k^3{v_{Ti}}^3}e^{-\frac{\omega^2}{2k^2{v_{Ti}}^2}}$$

Longitudinal waves: intermediate frequency range

propagation

$$v_{Ti} \ll \frac{\omega}{k} \ll v_{Te}$$

$$\varepsilon^l(\omega,k) = 1 + \frac{{\omega_{pe}}^2}{k^2{v_{Te}}^2}\left\{1 + i\sqrt{\frac{\pi}{2}}\frac{\omega}{kv_{Te}}\right\} - \frac{{\omega_{pi}}^2}{\omega^2}\left\{1 + 3\frac{k^2{v_{Ti}}^2}{\omega^2}\right\} + i\sqrt{\frac{\pi}{2}}\frac{\omega{\omega_{pi}}^2}{k^3{v_{Ti}}^3}e^{-\frac{\omega^2}{2k^2{v_{Ti}}^2}} = 0$$

$$\varepsilon^l(\omega,k) = 1 + \frac{{\omega_{pe}}^2}{k^2{v_{Te}}^2} - \frac{{\omega_{pi}}^2}{\omega^2}\left\{1 + 3\frac{k^2{v_{Ti}}^2}{\omega^2}\right\} = 0 \quad \Rightarrow \quad \omega^2 = \frac{{\omega_{pi}}^2}{1 + \frac{{\omega_{pe}}^2}{k^2{v_{Te}}^2}}\left\{1 + 3\frac{k^2{v_{Ti}}^2}{\omega^2}\right\}$$

$$\omega^2 = \frac{{\omega_{pi}}^2}{1 + \frac{1}{k^2{r_{De}}^2}}\left\{1 + 3k^2{r_{Di}}^2\left(1 + \frac{1}{k^2{r_{De}}^2}\right)\right\}$$

Longitudinal waves: intermediate frequency range

damping

$$v_{Ti} \ll \frac{\omega}{k} \ll v_{Te}$$

$$\varepsilon^l(\omega, k) = 1 + \frac{{\omega_{pe}}^2}{k^2{v_{Te}}^2}\left\{1 + i\sqrt{\frac{\pi}{2}}\frac{\omega}{kv_{Te}}\right\} - \frac{{\omega_{pi}}^2}{\omega^2}\left\{1 + 3\frac{k^2{v_{Ti}}^2}{\omega^2}\right\} + i\sqrt{\frac{\pi}{2}}\frac{\omega{\omega_{pi}}^2}{k^3{v_{Ti}}^3}e^{-\frac{\omega^2}{2k^2{v_{Ti}}^2}} = 0$$

$$Re\ \varepsilon^l(\omega, k) = 1 + \frac{{\omega_{pe}}^2}{k^2{v_{Te}}^2} - \frac{{\omega_{pi}}^2}{\omega^2}\left\{1 + 3\frac{k^2{v_{Ti}}^2}{\omega^2}\right\} \approx 1 + \frac{{\omega_{pe}}^2}{k^2{v_{Te}}^2} - \frac{{\omega_{pi}}^2}{\omega^2}$$

$$Im\ \varepsilon^l(\omega, k) = \sqrt{\frac{\pi}{2}}\left\{\frac{\omega{\omega_{pe}}^2}{k^3{v_{Te}}^3} + \frac{\omega{\omega_{pi}}^2}{k^3{v_{Ti}}^3}e^{-\frac{\omega^2}{2k^2{v_{Ti}}^2}}\right\}$$

$$\delta^l(k) = -\frac{Im\ \varepsilon^l(\omega, k)}{\frac{\partial Re\ \varepsilon^l(\omega, k)}{\partial\omega}}$$

Longitudinal waves: intermediate frequency range

damping

$$v_{Ti} \ll \frac{\omega}{k} \ll v_{Te}$$

$$Re\ \varepsilon^l(\omega,k) = 1 + \frac{{\omega_{pe}}^2}{k^2 {v_{Te}}^2} - \frac{{\omega_{pi}}^2}{\omega^2} \quad \Rightarrow \quad \frac{\partial Re\ \varepsilon^l(\omega,k)}{\partial\omega} = 2\frac{{\omega_{pi}}^2}{\omega^3}$$

$$Im\ \varepsilon^l(\omega,k) = \sqrt{\frac{\pi}{2}}\left\{\frac{\omega {\omega_{pe}}^2}{k^3 {v_{Te}}^3} + \frac{\omega {\omega_{pi}}^2}{k^3 {v_{Ti}}^3} e^{-\frac{\omega^2}{2k^2 {v_{Ti}}^2}}\right\}$$

$$\delta^l(k) = -\frac{Im\ \varepsilon^l(\omega,k)}{\frac{\partial Re\ \varepsilon^l(\omega,k)}{\partial\omega}} = -\sqrt{\frac{\pi}{8}}\frac{\omega^4}{k^3 {v_{Te}}^3}\frac{{\omega_{pe}}^2}{{\omega_{pi}}^2}\left\{1 + \frac{{\omega_{pi}}^2}{{\omega_{pe}}^2}\frac{{v_{Te}}^3}{{v_{Ti}}^3} e^{-\frac{\omega^2}{2k^2 {v_{Ti}}^2}}\right\}$$

$$e_i = Ze \qquad \frac{{\omega_{pe}}^2}{{\omega_{pi}}^2} = \frac{1}{Z}\frac{M}{m} \qquad \frac{{v_{Te}}^3}{{v_{Ti}}^3} = \left(\frac{MT_e}{mT_i}\right)^{3/2}$$

Longitudinal waves: intermediate frequency range

damping

$$v_{Ti} \ll \frac{\omega}{k} \ll v_{Te}$$

$$\delta^l(k) = -\frac{Im\, \varepsilon^l(\omega,k)}{\frac{\partial Re\, \varepsilon^l(\omega,k)}{\partial \omega}} = -\sqrt{\frac{\pi}{8}}\frac{\omega^4}{k^3 {v_{Te}}^3}\frac{{\omega_{pe}}^2}{{\omega_{pi}}^2}\left\{1+\frac{{\omega_{pi}}^2}{{\omega_{pe}}^2}\frac{{v_{Te}}^3}{{v_{Ti}}^3}e^{-\frac{\omega^2}{2k^2{v_{Ti}}^2}}\right\} =$$

$$= -\sqrt{\frac{\pi}{8}}\frac{M}{m}\frac{1}{Z}\frac{\omega^4}{k^3{v_{Te}}^3}\left\{1+Z\sqrt{\frac{M}{m}}\left(\frac{T_e}{T_i}\right)^{3/2} e^{-\frac{\omega^2}{2k^2{v_{Ti}}^2}}\right\}$$

$$e_i = Ze \qquad \frac{{\omega_{pe}}^2}{{\omega_{pi}}^2} = \frac{1}{Z}\frac{M}{m} \qquad \frac{{v_{Te}}^3}{{v_{Ti}}^3} = \left(\frac{MT_e}{mT_i}\right)^{3/2}$$

Longitudinal waves: intermediate frequency range

$$v_{Ti} \ll \frac{\omega}{k} \ll v_{Te}$$

propagation

$$\omega^2 = \frac{{\omega_{pi}}^2}{1 + \frac{1}{k^2 {r_{De}}^2}} \left\{ 1 + 3k^2 {r_{Di}}^2 \left(1 + \frac{1}{k^2 {r_{De}}^2} \right) \right\}$$

damping

$$\delta^l(k) = -\sqrt{\frac{\pi}{8}} \frac{M}{m} \frac{1}{Z} \frac{\omega^4}{k^3 {v_{Te}}^3} \left\{ 1 + Z\sqrt{\frac{M}{m}} \left(\frac{T_e}{T_i} \right)^{3/2} e^{-\frac{\omega^2}{2k^2 {v_{Ti}}^2}} \right\}$$

Longitudinal waves: intermediate frequency range

$$\omega^2 = \frac{\omega_{pi}{}^2}{1+\frac{1}{k^2 r_{De}{}^2}}\left\{1 + 3k^2 r_{Di}{}^2\left(1 + \frac{1}{k^2 r_{De}{}^2}\right)\right\}; \quad \delta^l(k) = -\sqrt{\frac{\pi}{8}\frac{M}{m}\frac{1}{Z}\frac{\omega^4}{k^3 v_{Te}{}^3}}\left\{1 + Z\sqrt{\frac{M}{m}}\left(\frac{T_e}{T_i}\right)^{3/2} e^{-\frac{\omega^2}{2k^2 v_{Ti}{}^2}}\right\}$$

$$v_{Ti} \ll \frac{\omega}{k} \ll v_{Te}$$

$$kr_{Di} \ll 1$$

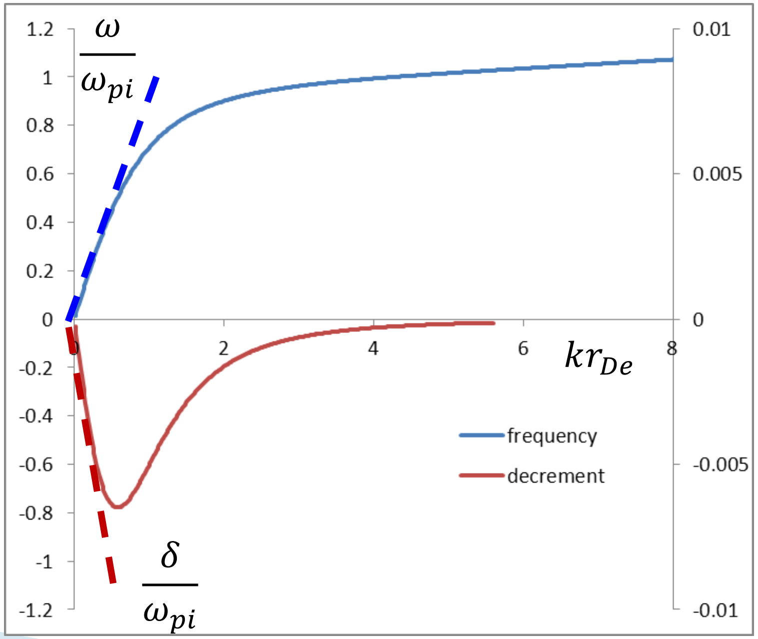
$$kr_{De} \ll 1$$

ion-acoustic oscillation

v_s - ion sound velocity

$$\omega^2 = k^2 r_{De}{}^2 \omega_{pi}{}^2\left\{1 + 3\frac{r_{Di}{}^2}{r_{De}{}^2}\right\} = k^2 Z\frac{\kappa T_e}{M}\left\{1 + 3\frac{T_i}{ZT_e}\right\} \rightarrow \boldsymbol{\omega = k v_s;\ v_s = \sqrt{Z\frac{\kappa T_e}{M}\left\{1 + 3\frac{T_i}{ZT_e}\right\}}}$$

$$\delta^l(k) = -\sqrt{\frac{\pi}{8}}\,\omega\sqrt{Z\frac{m}{M}}\left\{1 + Z\sqrt{\frac{M}{m}}\left(\frac{T_e}{T_i}\right)^{3/2} e^{-\frac{3}{2}-\frac{ZT_e}{2T_i}}\right\}$$



Longitudinal waves: intermediate frequency range

$$\omega^2 = \frac{\omega_{pi}^2}{1+\frac{1}{k^2 r_{De}^2}}\left\{1 + 3k^2 r_{Di}^2\left(1 + \frac{1}{k^2 r_{De}^2}\right)\right\}; \quad \delta^l(k) = -\sqrt{\frac{\pi}{8}\frac{M}{m}\frac{1}{Z}\frac{\omega^4}{k^3 v_{Te}^3}}\left\{1 + Z\sqrt{\frac{M}{m}}\left(\frac{T_e}{T_i}\right)^{3/2} e^{-\frac{\omega^2}{2k^2 v_{Ti}^2}}\right\}$$

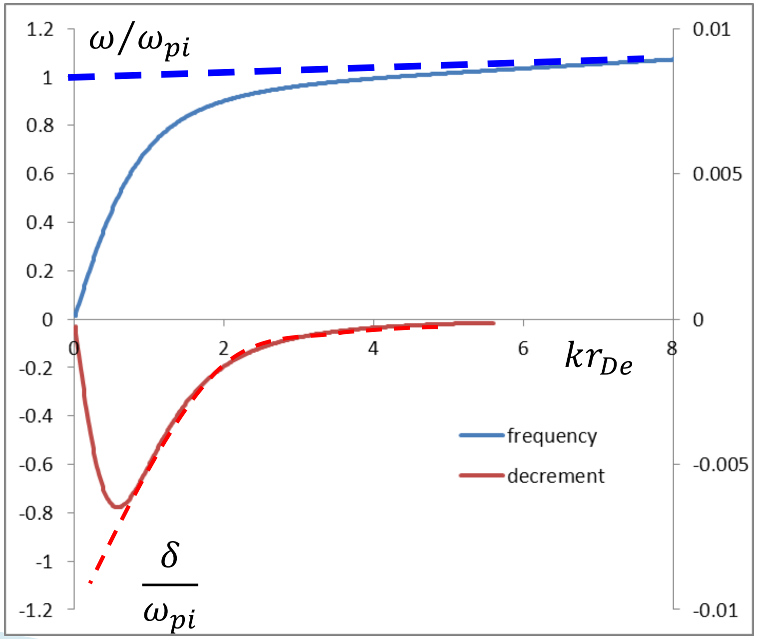
$$v_{Ti} \ll \frac{\omega}{k} \ll v_{Te}$$

$$k r_{Di} \ll 1$$

$$k r_{De} \gg 1$$

$$\omega^2 \approx \omega_{pi}^2$$

$$\delta^l(k) = -\sqrt{\frac{\pi}{8}}\,\omega\sqrt{Z\frac{m}{M}}\frac{\omega_{pi}}{k^3 r_{De}^3}\left\{1 + Z\sqrt{\frac{M}{m}}\left(\frac{T_e}{T_i}\right)^{3/2} e^{-\frac{\omega_{pi}^2}{2k^2 v_{Ti}^2}-\frac{3}{2}}\right\}$$



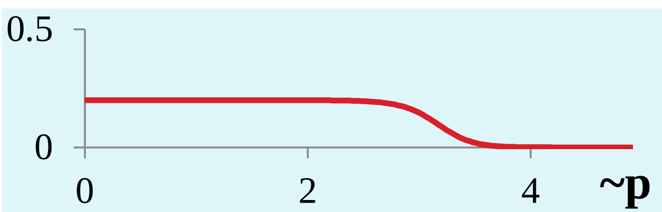
Dielectric permittivity for degenerate plasma

Homogeneous isotropic **degenerate** plasma:

$$\varepsilon^{l}(\omega,k) = 1 + \sum_{\alpha} \frac{{e_\alpha}^2}{\varepsilon_0 \omega k^2} \int d\vec{p}\, \frac{(\vec{k}\vec{v})^2}{\omega - \vec{k}\vec{v}} \left(\frac{\partial f_{0\alpha}}{\partial \mathcal{E}_\alpha}\right)$$

$$\varepsilon^{tr}(\omega,k) = 1 + \sum_{\alpha} \frac{{e_\alpha}^2}{2\varepsilon_0 \omega k^2} \int d\vec{p}\, \frac{[\vec{k}\times\vec{v}]^2}{\omega - \vec{k}\vec{v}} \left(\frac{\partial f_{0\alpha}}{\partial \mathcal{E}_\alpha}\right)$$

- Fermi distribution function $f_{0\alpha}(\mathcal{E}_\alpha) = f_{F\alpha} = \dfrac{\frac{2}{(2\pi\hbar)^3}}{exp\left(\frac{\mathcal{E}_\alpha - \mathcal{E}_{F\alpha}}{\kappa T_\alpha}\right)+1}$



$$\left(\frac{\partial f_{F\alpha}}{\partial \mathcal{E}_\alpha}\right) = -\frac{2}{(2\pi\hbar)^3}\delta(\mathcal{E}_\alpha - \mathcal{E}_{F\alpha})$$

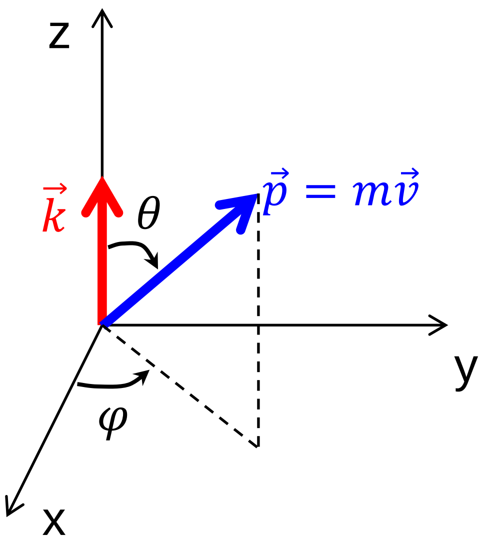
$$\mathcal{E}_\alpha = \frac{{p_\alpha}^2}{2m_\alpha}$$

$$\mathcal{E}_{F\alpha} = \frac{{p_{F\alpha}}^2}{2m_\alpha} = \frac{(3\pi^2)^{2/3}\hbar^2 {N_\alpha}^{2/3}}{2m_\alpha}$$

Longitudinal dielectric permittivity for homogeneous degenerate plasma

$$\varepsilon^l(\omega, k) = 1 + \sum_\alpha \frac{{e_\alpha}^2}{\varepsilon_0 \omega k^2} \int d\vec{p}\, \frac{\left(\vec{k}\vec{v}\right)^2}{\omega - \vec{k}\vec{v}} \left(\frac{\partial f_{0\alpha}}{\partial \mathcal{E}_\alpha}\right)$$

$$\left(\frac{\partial f_{F\alpha}}{\partial \mathcal{E}_\alpha}\right) = -\frac{2}{(2\pi\hbar)^3}\delta(\mathcal{E}_\alpha - \mathcal{E}_{F\alpha})$$



$$d\vec{p} = p^2 \sin\theta \;\, dp\, d\theta\, d\varphi$$

$$p = \sqrt{2m\mathcal{E}}$$

$$\vec{k}\vec{v} = k\sqrt{\frac{2\mathcal{E}}{m}}\cos\theta$$

$$d\vec{p} = \sqrt{\frac{2m^3}{\mathcal{E}}}\sin\theta \;\, d\mathcal{E}\, d\theta\, d\varphi = -\sqrt{\frac{2m^3}{\mathcal{E}}}\, d\mathcal{E}\, d(\cos\theta)\, d\varphi$$

$$\mathcal{E} \to \mathcal{E}_{F\alpha}$$
$$v \to v_{F\alpha}$$

$$\int_{-1}^{1} \frac{t^2 dt}{\omega - k v_{F\alpha} t}$$

$$2\pi$$

$$v_{F\alpha} = \frac{p_{F\alpha}}{m_\alpha} = \frac{(3\pi^2)^{1/3}\hbar {N_\alpha}^{1/3}}{m_\alpha}$$

Longitudinal dielectric permittivity for homogeneous degenerate plasma

$t = \cos\theta$ → not time!

$$\varepsilon^{l}(\omega, k) = 1 + \frac{4\pi}{(2\pi\hbar)^3}\sum_{\alpha}\frac{{e_\alpha}^2}{\varepsilon_0 \omega k^2}\sqrt{\frac{2m^3}{\mathcal{E}_{F\alpha}}}\int_{-1}^{1}\frac{(kv_{F\alpha})^2 t^2 dt}{\omega - kv_{F\alpha}t}$$

$$\varepsilon^{l}(\omega, k) = 1 + \sum_{\alpha}\frac{3{\omega_{p\alpha}}^2}{k^2 {v_{F\alpha}}^2}\left[1 - \frac{1}{2}\frac{\omega}{kv_{F\alpha}}\ln\frac{\omega + kv_{F\alpha}}{\omega - kv_{F\alpha}}\right]$$

$$v_{F\alpha} = \frac{(3\pi^2)^{1/3}\hbar {N_\alpha}^{1/3}}{m_\alpha}$$

Transverse dielectric permittivity for homogeneous degenerate plasma

$$\varepsilon^{tr}(\omega,k) = 1 + \sum_{\alpha} \frac{{e_\alpha}^2}{2\varepsilon_0 \omega k^2} \int d\vec{p}\, \frac{\left[\vec{k}\times\vec{v}\right]^2}{\omega - \vec{k}\vec{v}} \left(\frac{\partial f_{0\alpha}}{\partial \mathcal{E}_\alpha}\right)$$

$$\varepsilon^{tr}(\omega,k) = 1 - \sum_{\alpha} \frac{3{\omega_{p\alpha}}^2}{2\omega^2}\left[1 - \left(1 - \frac{\omega^2}{k^2 {v_{F\alpha}}^2}\right)\cdot\left(1 - \frac{1}{2}\frac{\omega}{k v_{F\alpha}} \ln\frac{\omega + k v_{F\alpha}}{\omega - k v_{F\alpha}}\right)\right]$$

$$v_{F\alpha} = \frac{(3\pi^2)^{1/3}\hbar {N_\alpha}^{1/3}}{m_\alpha}$$

Dielectric permittivity for homogeneous degenerate plasma

$$\varepsilon^{l}(\omega,k) = 1 + \sum_{\alpha} \frac{3{\omega_{p\alpha}}^2}{k^2{v_{F\alpha}}^2}\left[1 - \frac{1}{2}\frac{\omega}{kv_{F\alpha}}\ln\frac{\omega + kv_{F\alpha}}{\omega - kv_{F\alpha}}\right]$$

$$\varepsilon^{tr}(\omega,k) = 1 - \sum_{\alpha} \frac{3{\omega_{p\alpha}}^2}{2\omega^2}\left[1 - \left(1 - \frac{\omega^2}{k^2{v_{F\alpha}}^2}\right)\cdot\left(1 - \frac{1}{2}\frac{\omega}{kv_{F\alpha}}\ln\frac{\omega + kv_{F\alpha}}{\omega - kv_{F\alpha}}\right)\right]$$

$$\varepsilon^{l}(\omega,k) = 0$$

$$\left[k^2 - \frac{\omega^2}{c^2}\varepsilon^{tr}(\omega,k)\right]^2 = 0$$

Longitudinal $\vec{E}||\vec{k}$

Transverse $\vec{E}\perp\vec{k}$

Waves in the degenerate plasma

$\ln(\omega - kv_{F\alpha}) = \ln|\omega - kv_{F\alpha}| - i\pi\delta(\omega - kv_{F\alpha})$ only if $\omega < kv_{F\alpha}$!

Longitudinal dielectric permittivity for homogeneous degenerate plasma: static limit

$$\varepsilon^l(\omega,k) = 1 + \sum_\alpha \frac{3{\omega_{p\alpha}}^2}{k^2{v_{F\alpha}}^2}\left[1 - \frac{1}{2}\frac{\omega}{kv_{F\alpha}}\ln\frac{\omega + kv_{F\alpha}}{\omega - kv_{F\alpha}}\right]$$

Static limit $\omega \to 0$ ⇨ $\omega \cdot \ln(\omega - kv_{F\alpha}) \to \omega \cdot (\ln|\omega - kv_{F\alpha}| - i\pi\delta(\omega - kv_{F\alpha})) \to 0$

$$\varepsilon^l(\omega,k) = 1 + \sum_\alpha \frac{3{\omega_{p\alpha}}^2}{k^2{v_{F\alpha}}^2} = 1 + \frac{1}{k^2{r_D}^2}$$

$$\frac{1}{{r_D}^2} = \sum_\alpha \frac{1}{{r_{D\alpha}}^2}$$

$$r_{D\alpha} = \frac{v_{F\alpha}}{\sqrt{3}\omega_{p\alpha}}$$

$\ln|\omega - kv_{F\alpha}| - i\pi\delta(\omega - kv_{F\alpha})$ only if $\omega < kv_{F\alpha}$!

$$v_{F\alpha} = \frac{(3\pi^2)^{1/3}\hbar {N_\alpha}^{1/3}}{m_\alpha}$$

High frequency plasma waves in the collisionless degenerate plasma

$$\varepsilon^l(\omega,k) = 1 + \sum_\alpha \frac{3{\omega_{p\alpha}}^2}{k^2 {v_{F\alpha}}^2}\left[1 - \frac{1}{2}\frac{\omega}{k v_{F\alpha}} \ln\frac{\omega + k v_{F\alpha}}{\omega - k v_{F\alpha}}\right]$$

$$\varepsilon^l(\omega,k) = 0$$

$$\frac{\omega}{k} \gg v_{Fe}, v_{Fi}$$

$$x = \frac{k v_{Fe}}{\omega} \ll 1$$

$$\ln(1+x) \approx x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5}$$

$$\ln\frac{1+x}{1-x} = \ln(1+x) - \ln(1-x) \approx x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \left(x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5}\right) =$$

$$= 2x \cdot \left(1 + \frac{x^2}{3} + \frac{x^4}{5}\right)$$

$$\left[\ldots - \ldots\right] = 1 - \frac{1}{2x}\ln\frac{1+x}{1-x} = -\frac{x^2}{3} - \frac{x^4}{5}$$

$$\varepsilon^l(\omega,k) = 1 - \sum_\alpha \frac{3{\omega_{p\alpha}}^2}{\omega^2}\left[\frac{1}{3} + \frac{x^2}{5}\right]$$

High frequency plasma waves in the collisionless degenerate plasma

$$\varepsilon^{l}(\omega,k) = 1 + \sum_{\alpha} \frac{3{\omega_{p\alpha}}^2}{k^2 {v_{F\alpha}}^2}\left[1 - \frac{1}{2}\frac{\omega}{k v_{F\alpha}} \ln \frac{\omega + k v_{F\alpha}}{\omega - k v_{F\alpha}}\right]$$

$$\varepsilon^{l}(\omega,k) = 0$$

$$\frac{\omega}{k} \gg v_{Fe}, v_{Fi}$$

$$r_{D\alpha} = \frac{v_{F\alpha}}{\sqrt{3}\omega_{p\alpha}}$$

$$\varepsilon^{l}(\omega,k) = 1 - \sum_{\alpha} \frac{{\omega_{p\alpha}}^2}{\omega^2}\left[1 + \frac{3}{5}\frac{k^2 {v_{F\alpha}}^2}{{\omega_{p\alpha}}^2}\right] = 0$$

$$\boldsymbol{\omega^2 = {\omega_{pe}}^2 \left[1 + \frac{9}{5} k^2 {r_{De}}^2\right]}$$

⇔

Maxwellian plasma:

$$\omega^2 = {\omega_{pe}}^2\left(1 + 3k^2 {r_{De}}^2\right)$$

$$\delta(k) = -\sqrt{\frac{\pi}{8}} \cdot \frac{\omega_{pe}}{k^3 {r_{De}}^3} \cdot e^{-\frac{1}{2k^2 {r_{De}}^2} - \frac{3}{2}}$$

Landau Damping

But: no damping for degenerate plasma!

High frequency plasma waves in the collisionless degenerate plasma: $k^2 {r_{De}}^2 \gg 1$

$$\varepsilon^l(\omega,k) = 1 + \sum_\alpha \frac{3{\omega_{p\alpha}}^2}{k^2 {v_{F\alpha}}^2}\left[1 - \frac{1}{2}\frac{\omega}{kv_{F\alpha}}\ln\frac{\omega + kv_{F\alpha}}{\omega - kv_{F\alpha}}\right]$$

$$\varepsilon^l(\omega,k) = 0$$

$$\frac{\omega}{k} \gg v_{Fe}, v_{Fi}$$

$$r_{D\alpha} = \frac{v_{F\alpha}}{\sqrt{3}\omega_{p\alpha}}$$

$$\varepsilon^l(\omega,k) = 1 + \frac{1}{k^2 {r_{De}}^2}\left[1 - \frac{1}{2}\frac{\omega}{kv_{F\alpha}}\ln\frac{\omega + kv_{F\alpha}}{\omega - kv_{F\alpha}}\right] = 0$$

$$k^2 {r_{De}}^2 + 1 - \frac{1}{2}\frac{\omega}{kv_{F\alpha}}\ln\frac{\omega + kv_{F\alpha}}{\omega - kv_{F\alpha}} = 0$$

$$B = \frac{\omega}{kv_{F\alpha}}$$

$$2k^2 {r_{De}}^2 + 2 = B\ln\frac{B+1}{B-1} \gg 1 \quad \Rightarrow \quad B \sim 1 + ..$$

$$B = 1 + \varepsilon$$

$$e^{-2k^2 {r_{De}}^2 - 2} = \left(\frac{\varepsilon}{2+\varepsilon}\right)^{1+\varepsilon} \approx \frac{\varepsilon}{2}$$

High frequency plasma waves in the collisionless degenerate plasma: $k^2 {r_{De}}^2 \gg 1$

$$\varepsilon^l(\omega, k) = 1 + \sum_\alpha \frac{3{\omega_{p\alpha}}^2}{k^2 {v_{F\alpha}}^2}\left[1 - \frac{1}{2}\frac{\omega}{k v_{F\alpha}} \ln \frac{\omega + k v_{F\alpha}}{\omega - k v_{F\alpha}}\right]$$

$$\varepsilon^l(\omega, k) = 0$$

$$\frac{\omega}{k} \gg v_{Fe}, v_{Fi}$$

$$r_{D\alpha} = \frac{v_{F\alpha}}{\sqrt{3}\omega_{p\alpha}}$$

$$B = \frac{\omega}{k v_{F\alpha}}$$

$$B = 1 + \varepsilon$$

$$e^{-2k^2 {r_{De}}^2 - 2} = \left(\frac{\varepsilon}{2+\varepsilon}\right)^{1+\varepsilon} \approx \frac{\varepsilon}{2}$$

$$\omega = k v_{F\alpha}\left(1 + 2e^{-2k^2 {r_{De}}^2 - 2}\right)$$

"zero-point sound"

High frequency plasma waves in the collisionless degenerate plasma

$$\varepsilon^{l}(\omega,k)=1+\frac{3{\omega_{p\alpha}}^{2}}{k^{2}{v_{Fe}}^{2}}\left[1-\frac{1}{2}\frac{\omega}{kv_{Fe}}\ln\frac{\omega+kv_{Fe}}{\omega-kv_{Fe}}\right]=0$$

$$\omega^{2}={\omega_{pe}}^{2}\left[1+\frac{9}{5}k^{2}{r_{De}}^{2}\right]$$

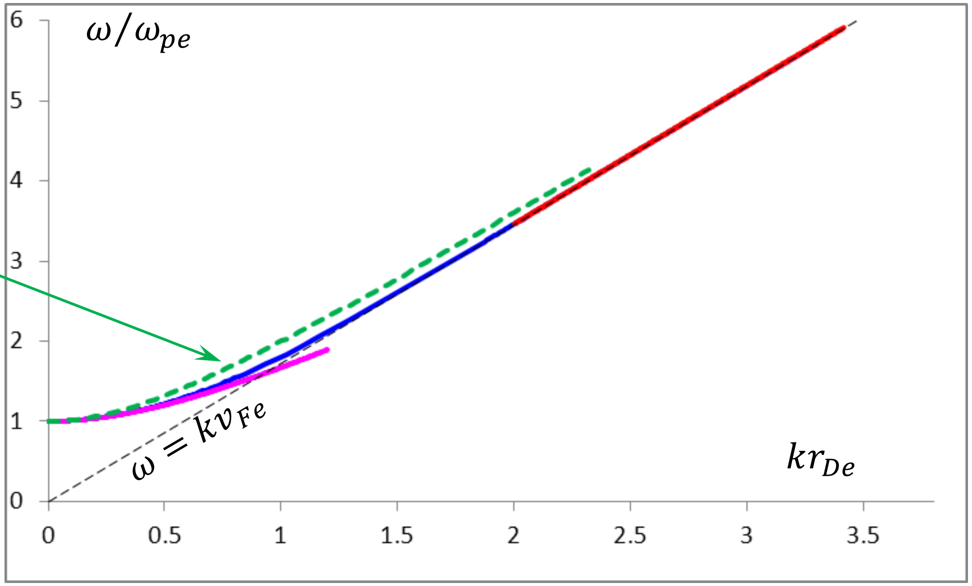
$$\omega=kv_{F\alpha}\left(1+2e^{-2k^{2}{r_{De}}^{2}-2}\right)$$

"zero-point sound"

Maxwellian plasma

$\omega^{2}={\omega_{pe}}^{2}(1+3k^{2}{r_{De}}^{2})$

Landau Damping



Summary

- Waves in homogeneous Maxwellian plasma:
  - High frequency transverse waves $\omega^2 = {\omega_{pe}}^2 + k^2c^2$
  - Short wavelength longitudinal oscillations (strongly damped) $\frac{\omega}{kv_{Te}} = \frac{\pi}{\sqrt{2\ln\left(k^2{r_{De}}^2\right)}} - i\sqrt{2\ln\left(k^2{r_{De}}^2\right)}$
  - Longitudinal waves: intermediate frequency range ($v_{Ti} \ll \frac{\omega}{k} \ll v_{Te}$):
    - ion-acoustic waves: $\omega = kv_s$; $v_s = \sqrt{Z\frac{\kappa T_e}{M}\left\{1 + 3\frac{T_i}{ZT_e}\right\}}$, damping
    - short wavelength: $\omega \approx \omega_{pi}$, damping

- Dielectric permittivity for degenerate plasma
  - Longitudinal $\varepsilon^l(\omega, k) = 1 + \sum_\alpha \frac{3{\omega_{p\alpha}}^2}{k^2{v_{F\alpha}}^2}\left[1 - \frac{1}{2}\frac{\omega}{kv_{F\alpha}}\ln\frac{\omega+kv_{F\alpha}}{\omega-kv_{F\alpha}}\right]$
  - Transverse $\varepsilon^{tr}(\omega, k) = 1 - \sum_\alpha \frac{3{\omega_{p\alpha}}^2}{2\omega^2}\left[1 - \left(1 - \frac{\omega^2}{k^2{v_{F\alpha}}^2}\right)\cdot\left(1 - \frac{1}{2}\frac{\omega}{kv_{F\alpha}}\ln\frac{\omega+kv_{F\alpha}}{\omega-kv_{F\alpha}}\right)\right]$

- Longitudinal waves in degenerate plasma
  - Debye shielding $r_{D\alpha} = \frac{v_{F\alpha}}{\sqrt{3}\omega_{p\alpha}}$
  - High frequency longitudinal plasma waves $\omega^2 = {\omega_{pe}}^2\left[1 + \frac{9}{5}k^2{r_{De}}^2\right]$ → no damping!
  - High frequency short wavelength longitudinal plasma oscillations:

$$\omega = kv_{F\alpha}\left(1 + 2e^{-2k^2{r_{De}}^2-2}\right)$$ → “zero-point sound”

- Next:
  - Ion-acoustic waves in degenerate plasma
  - Transverse high frequency waves in degenerate plasma, skin-effect
  - General classification of waves in collisionless plasma
    - Kinetic approach with paqrticle collisions