Waves and oscillations in the collisionless plasma (summary)

M.Krasilnikov
"Electrodynamics of plasma and beams“ – 2012/2013
L#9

Discussed last time…

- Waves in homogeneous Maxwellian plasma:
  - High frequency transverse waves $\omega^2 = {\omega_{pe}}^2 + k^2c^2$
  - Short wavelength longitudinal oscillations (strongly damped) $\frac{\omega}{kv_{Te}} = \frac{\pi}{\sqrt{2\ln\left(k^2{r_{De}}^2\right)}} - i\sqrt{2\ln\left(k^2{r_{De}}^2\right)}$
  - Longitudinal waves: intermediate frequency range ($v_{Ti} \ll \frac{\omega}{k} \ll v_{Te}$):
    - ion-acoustic waves: $\omega = kv_s$; $v_s = \sqrt{Z\frac{\kappa T_e}{M}\left\{1 + 3\frac{T_i}{ZT_e}\right\}}$, damping
    - short wavelength:$\omega \approx \omega_{pi}$, damping

- Dielectric permittivity for degenerate plasma
  - Longitudinal $\varepsilon^l(\omega, k\,) = 1 + \sum_\alpha \frac{3{\omega_{p\alpha}}^2}{k^2{v_{F\alpha}}^2}\left[1 - \frac{1}{2}\frac{\omega}{kv_{F\alpha}}\ln\frac{\omega+kv_{F\alpha}}{\omega-kv_{F\alpha}}\right]$
  - Transverse $\varepsilon^{tr}(\omega, k\,) = 1 - \sum_\alpha \frac{3{\omega_{p\alpha}}^2}{2\omega^2}\left[1 - \left(1 - \frac{\omega^2}{k^2{v_{F\alpha}}^2}\right)\cdot\left(1 - \frac{1}{2}\frac{\omega}{kv_{F\alpha}}\ln\frac{\omega+kv_{F\alpha}}{\omega-kv_{F\alpha}}\right)\right]$

- Longitudinal waves in degenerate plasma
  - Debye shielding $r_{D\alpha} = \frac{v_{F\alpha}}{\sqrt{3}\omega_{p\alpha}}$
  - High frequency longitudinal plasma waves $\omega^2 = {\omega_{pe}}^2\left[1 + \frac{9}{5}k^2{r_{De}}^2\right]$ → no damping!
  - High frequency short wavelength longitudinal plasma oscillations:

$$\omega = kv_{F\alpha}\left(1 + 2e^{-2k^2{r_{De}}^2-2}\right)$$ → "zero-point sound"

- Not yet discussed:
  - Ion-acoustic waves in degenerate plasma
  - Transverse high frequency waves in degenerate plasma, skin-effect

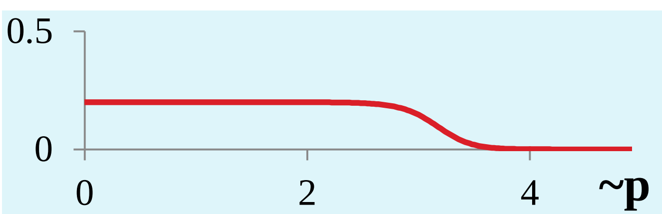
Dielectric permittivity for degenerate plasma

Homogeneous isotropic **degenerate** plasma:

$$\varepsilon^{l}(\omega,k) = 1 + \sum_{\alpha} \frac{{e_\alpha}^2}{\varepsilon_0 \omega k^2} \int d\vec{p}\, \frac{\left(\vec{k}\vec{v}\right)^2}{\omega - \vec{k}\vec{v}} \left(\frac{\partial f_{0\alpha}}{\partial \mathcal{E}_\alpha}\right)$$

$$\varepsilon^{tr}(\omega,k) = 1 + \sum_{\alpha} \frac{{e_\alpha}^2}{2\varepsilon_0 \omega k^2} \int d\vec{p}\, \frac{\left[\vec{k}\times\vec{v}\right]^2}{\omega - \vec{k}\vec{v}} \left(\frac{\partial f_{0\alpha}}{\partial \mathcal{E}_\alpha}\right)$$

- Fermi distribution function $f_{0\alpha}(\mathcal{E}_\alpha) = f_{F\alpha} = \dfrac{\frac{2}{(2\pi\hbar)^3}}{exp\left(\frac{\mathcal{E}_\alpha - \mathcal{E}_{F\alpha}}{\kappa T_\alpha}\right)+1}$



$$\left(\frac{\partial f_{F\alpha}}{\partial \mathcal{E}_\alpha}\right) = -\frac{2}{(2\pi\hbar)^3}\,\delta(\mathcal{E}_\alpha - \mathcal{E}_{F\alpha})$$

$$\mathcal{E}_\alpha = \frac{{p_\alpha}^2}{2m_\alpha}$$

$$\mathcal{E}_{F\alpha} = \frac{{p_{F\alpha}}^2}{2m_\alpha} = \frac{(3\pi^2)^{2/3}\hbar^2 {N_\alpha}^{2/3}}{2m_\alpha}$$

Dielectric permittivity for homogeneous **degenerate** plasma

$$\varepsilon^{l}(\omega,k) = 1 + \sum_{\alpha} \frac{3{\omega_{p\alpha}}^{2}}{k^{2}{v_{F\alpha}}^{2}} \left[1 - \frac{1}{2}\frac{\omega}{k v_{F\alpha}} \ln \frac{\omega + k v_{F\alpha}}{\omega - k v_{F\alpha}} \right]$$

$$\varepsilon^{tr}(\omega,k) = 1 - \sum_{\alpha} \frac{3{\omega_{p\alpha}}^{2}}{2\omega^{2}} \left[1 - \left(1 - \frac{\omega^{2}}{k^{2}{v_{F\alpha}}^{2}} \right) \cdot \left(1 - \frac{1}{2}\frac{\omega}{k v_{F\alpha}} \ln \frac{\omega + k v_{F\alpha}}{\omega - k v_{F\alpha}} \right) \right]$$

$$\varepsilon^{l}(\omega,k) = 0$$

Longitudinal $\vec{E}||\vec{k}$

$$\left[k^{2} - \frac{\omega^{2}}{c^{2}} \varepsilon^{tr}(\omega,k) \right]^{2} = 0$$

Transverse $\vec{E} \perp \vec{k}$

Waves in the degenerate plasma

$\ln(\omega - k v_{F\alpha}) = \ln|\omega - k v_{F\alpha}| - i\pi\delta(\omega - k v_{F\alpha})$ only if $\omega < k v_{F\alpha}$!

Longitudinal waves in degenerate plasma: intermediate frequency (phase velocity) range

$$\varepsilon^l(\omega,k) = 1 + \sum_\alpha \frac{3{\omega_{p\alpha}}^2}{k^2 {v_{F\alpha}}^2}\left[1 - \frac{1}{2}\frac{\omega}{k v_{F\alpha}} \ln\frac{\omega + k v_{F\alpha}}{\omega - k v_{F\alpha}}\right]$$

$$\varepsilon^l(\omega,k) = 0$$

$$v_{Fi} \ll \frac{\omega}{k} \ll v_{Fe}$$

$$x = \frac{k v_{Fi}}{\omega} \ll 1$$

$$\ln(1+x) \approx x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5}$$

$$\ln\frac{1+x}{1-x} = \ln(1+x) - \ln(1-x) \approx x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \left(x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5}\right) =$$

$$= 2x \cdot \left(1 + \frac{x^2}{3} + \frac{x^4}{5}\right)$$

$$\left[\ldots - \ldots\right] = 1 - \frac{1}{2x}\ln\frac{1+x}{1-x} = -\frac{x^2}{3} - \frac{x^4}{5}$$

$$\delta{\varepsilon^l}_i(\omega,k) = -\frac{3{\omega_{pi}}^2}{\omega^2}\left[\frac{1}{3} + \frac{x^2}{5}\right] \approx -\frac{{\omega_{pi}}^2}{\omega^2}$$

Longitudinal waves in degenerate plasma: intermediate frequency (phase velocity) range

$$\varepsilon^l(\omega,k) = 1 + \sum_\alpha \frac{3{\omega_{p\alpha}}^2}{k^2{v_{F\alpha}}^2}\left[1 - \frac{1}{2}\frac{\omega}{kv_{F\alpha}}\ln\frac{\omega + kv_{F\alpha}}{\omega - kv_{F\alpha}}\right]$$

$$\varepsilon^l(\omega,k) = 0$$

$$v_{Fi} \ll \frac{\omega}{k} \ll v_{Fe}$$

$$x = \frac{\omega}{kv_{Fe}} \ll 1$$

$\ln(\omega - kv_{F\alpha}) = \ln|\omega - kv_{F\alpha}| - i\pi\delta(\omega - kv_{F\alpha})$ only if $\omega < kv_{F\alpha}$!

$$\delta{\varepsilon^l}_e(\omega,k) = \frac{3{\omega_{pe}}^2}{k^2{v_{Fe}}^2}\left[1 + i\frac{\pi}{2}\frac{\omega}{kv_{Fe}}\right]$$

Longitudinal waves in degenerate plasma: intermediate frequency (phase velocity) range

$$\varepsilon^l(\omega,k) = 1 + \sum_\alpha \frac{3{\omega_{p\alpha}}^2}{k^2 {v_{F\alpha}}^2}\left[1 - \frac{1}{2}\frac{\omega}{k v_{F\alpha}} \ln \frac{\omega + k v_{F\alpha}}{\omega - k v_{F\alpha}}\right]$$

$$\varepsilon^l(\omega,k) = 0$$

$$v_{Fi} \ll \frac{\omega}{k} \ll v_{Fe}$$

$$\delta {\varepsilon^l}_i(\omega,k) = -\frac{3{\omega_{pi}}^2}{\omega^2}\left[\frac{1}{3} + \frac{x^2}{5}\right] \approx -\frac{{\omega_{pi}}^2}{\omega^2}$$

$$\delta {\varepsilon^l}_e(\omega,k) = \frac{3{\omega_{pe}}^2}{k^2 {v_{Fe}}^2}\left[1 + i\frac{\pi}{2}\frac{\omega}{k v_{Fe}}\right]$$

$$\varepsilon^l(\omega,k) = 1 + \frac{3{\omega_{pe}}^2}{k^2 {v_{Fe}}^2}\left[1 + i\frac{\pi}{2}\frac{\omega}{k v_{Fe}}\right] - \frac{{\omega_{pi}}^2}{\omega^2}$$

Longitudinal waves in degenerate plasma: intermediate frequency (phase velocity) range

$$\varepsilon^l(\omega, k) = 1 + \frac{3\omega_{pe}^2}{k^2 v_{Fe}^2}\left[1 + i\frac{\pi}{2}\frac{\omega}{k v_{Fe}}\right] - \frac{\omega_{pi}^2}{\omega^2} = 0$$

$$v_{Fi} \ll \frac{\omega}{k} \ll v_{Fe}$$

$$Re\,\varepsilon^l(\omega, k) = 1 + \frac{3\omega_{pe}^2}{k^2 v_{Fe}^2} - \frac{\omega_{pi}^2}{\omega^2} = 0$$

propagation

$$\boldsymbol{\omega^2 = \frac{\omega_{pi}^2}{1 + \frac{3\omega_{pe}^2}{k^2 v_{Fe}^2}}}$$

$$Im\,\varepsilon^l(\omega, k) = i\frac{3\pi}{2}\frac{\omega_{pe}^2\omega}{k^3 v_{Fe}^3}$$

damping

$$\delta^l(k) = -\frac{Im\,\varepsilon^l(\omega, k)}{\frac{\partial Re\,\varepsilon^l(\omega, k)}{\partial\omega}} = -\frac{3\pi}{4}\frac{\omega^4}{k^3 v_{Fe}^3}\frac{\omega_{pe}^2}{\omega_{pi}^2}$$

$$\boldsymbol{\delta^l(k) = -\frac{3\pi}{4}\frac{\omega^4}{k^3 v_{Fe}^3}\cdot\frac{M}{m}}$$

NB: for non-degenerate plasma

$$\delta^l(k) = -\sqrt{\frac{\pi}{8}}\frac{M}{m}\frac{1}{Z}\frac{\omega^4}{k^3 v_{Te}^3}\left\{1 + Z\sqrt{\frac{M}{m}}\left(\frac{T_e}{T_i}\right)^{3/2} e^{-\frac{\omega^2}{2k^2 v_{Ti}^2}}\right\}$$

Longitudinal waves in partially degenerate plasma: intermediate frequency (phase velocity) range

$$\omega^2 = \frac{{\omega_{pi}}^2}{1 + \frac{3{\omega_{pe}}^2}{k^2 {v_{Fe}}^2}}$$

$$\delta^l(k) = -\frac{3\pi}{4}\frac{\omega^4}{k^3 {v_{Fe}}^3}\cdot\frac{M}{m}$$

NB: for non-degenerate plasma

$$\delta^l(k) = -\sqrt{\frac{\pi}{8}}\frac{M}{m}\frac{1}{Z}\frac{\omega^4}{k^3 {v_{Te}}^3}\left\{1 + Z\sqrt{\frac{M}{m}}\left(\frac{T_e}{T_i}\right)^{3/2} e^{-\frac{\omega^2}{2k^2 {v_{Ti}}^2}}\right\}$$

nondegenerate ⟵ $v_{Ti} \ll \frac{\omega}{k} \ll v_{Fe}$ ⟶ degenerate

$$\boldsymbol{\delta^l(k) = -\frac{3\pi}{4}\frac{\omega^4}{k^3 {v_{Fe}}^3}\cdot\frac{M}{m} - \sqrt{\frac{\pi}{8}}\frac{\omega^4}{k^3 {v_{Ti}}^3} e^{-\frac{\omega^2}{2k^2 {v_{Ti}}^2}}}$$

Ion acoustic waves in degenerate plasma

$$\boldsymbol{\omega^2 = \frac{{\omega_{pi}}^2}{1 + \frac{3{\omega_{pe}}^2}{k^2 {v_{Fe}}^2}}}$$

$$v_{Fi} \ll \frac{\omega}{k} \ll v_{Fe}$$

$$\frac{k^2 {v_{Fe}}^2}{3{\omega_{pe}}^2} \ll 1$$

$$\boldsymbol{\omega = k v_s; \;\; v_s = \frac{v_{Fe}}{\sqrt{3}}\sqrt{\frac{m}{M}}}$$

To compare: nondegenerate plasma: $\omega = k v_s;\ v_s \approx \sqrt{Z\frac{\kappa T_e}{M}}$

$$v_{F\alpha} = \frac{(3\pi^2)^{1/3}\hbar {N_\alpha}^{1/3}}{m_\alpha}$$

Ion acoustic waves in degenerate plasma

$$\varepsilon^l(\omega,k) = 1 + \sum_{\alpha} \frac{3{\omega_{p\alpha}}^2}{k^2{v_{F\alpha}}^2}\left[1 - \frac{1}{2}\frac{\omega}{kv_{F\alpha}}\ln\frac{\omega + kv_{F\alpha}}{\omega - kv_{F\alpha}}\right]$$

$$\frac{\omega}{k} \ll v_{Fe}$$

$$\frac{\omega}{k} \rightarrow v_{Fi}$$

$$\frac{kv_{Fi}}{\omega_{pi}} \gg 1$$

$$\varepsilon^l(\omega,k) = 1 + \frac{3{\omega_{pe}}^2}{k^2{v_{Fe}}^2}\left[1 + i\frac{\pi}{2}\frac{\omega}{kv_{Fe}}\right] + \frac{3{\omega_{pi}}^2}{k^2{v_{Fi}}^2}\left[1 - \frac{1}{2}\frac{\omega}{kv_{Fi}}\ln\frac{\omega+kv_{Fi}}{\omega-kv_{Fi}}\right] = 0$$

$$\frac{k^2{v_{Fi}}^2}{3{\omega_{pi}}^2} + \frac{{v_{Fi}}^2}{{v_{Fe}}^2}\cdot\frac{M}{m}\left[1 + i\frac{\pi}{2}\frac{\omega}{kv_{Fe}}\right] + 1 - \frac{1}{2}\frac{\omega}{kv_{Fi}}\ln\frac{\omega+kv_{Fi}}{\omega-kv_{Fi}} = 0$$

$$\omega \sim kv_{Fi}$$

$$\frac{k^2{v_{Fi}}^2}{3{\omega_{pi}}^2} + i\frac{\pi}{2}\frac{{v_{Fi}}^3}{{v_{Fe}}^3}\cdot\frac{M}{m} + 1 = \frac{1}{2}\ln\frac{\omega+kv_{Fi}}{\omega-kv_{Fi}}$$

$$\frac{\omega - kv_{Fi}}{\omega + kv_{Fi}} = \exp\left[-\frac{2k^2{v_{Fi}}^2}{3{\omega_{pi}}^2} - i\pi\frac{{v_{Fi}}^3}{{v_{Fe}}^3}\cdot\frac{M}{m} - 2\right]$$

Ion acoustic waves in degenerate plasma

$\frac{\omega}{k} \ll v_{Fe}$ $\frac{\omega}{k} \to v_{Fi}$ $\frac{kv_{Fi}}{\omega_{pi}} \gg 1$ $\omega \sim kv_{Fi}$

$$\varepsilon^l(\omega, k) = 1 + \frac{3\omega_{pe}^2}{k^2 v_{Fe}^2}\left[1 + i\frac{\pi}{2}\frac{\omega}{kv_{Fe}}\right] + \frac{3\omega_{pi}^2}{k^2 v_{Fi}^2}\left[1 - \frac{1}{2}\frac{\omega}{kv_{Fi}}\ln\frac{\omega + kv_{Fi}}{\omega - kv_{Fi}}\right] = 0$$

$$\frac{\omega - kv_{Fi}}{\omega + kv_{Fi}} = \exp\left[-\frac{2k^2 v_{Fi}^2}{3\omega_{pi}^2} - i\pi\frac{v_{Fi}^3}{v_{Fe}^3}\cdot\frac{M}{m} \; -2\right]$$

$$\omega - kv_{Fi} \approx 2kv_{Fi}\exp\left[-\frac{2k^2 v_{Fi}^2}{3\omega_{pi}^2} - i\pi\frac{v_{Fi}^3}{v_{Fe}^3}\cdot\frac{M}{m} \; -2\right]$$

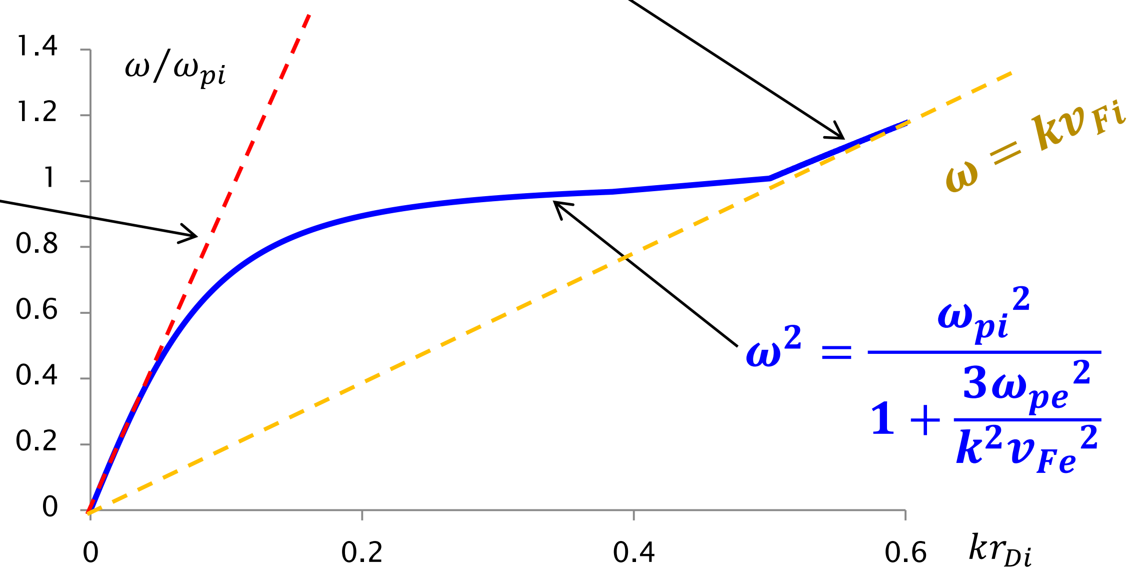
$$\boldsymbol{\omega = kv_{Fi}\left\{1 + 2\cdot\left(1 - i\pi\frac{v_{Fi}^3}{v_{Fe}^3}\cdot\frac{M}{m}\right)\cdot exp\left[-\frac{2k^2 v_{Fi}^2}{3\omega_{pi}^2} \; -2\right]\right\}}$$

Ion acoustic waves in degenerate plasma

$$\omega = kv_{Fi}\left\{1 + 2\,exp\left[-\frac{2k^2 {v_{Fi}}^2}{3{\omega_{pi}}^2} - 2\right]\right\}$$

$$\omega = kv_s$$

$$v_s = \frac{v_{Fe}}{\sqrt{3}}\sqrt{\frac{m}{M}}$$



$$\omega^2 = \frac{{\omega_{pi}}^2}{1 + \frac{3{\omega_{pe}}^2}{k^2 {v_{Fe}}^2}}$$

damping

$$\delta^l(k) = -\frac{3\pi}{4}\frac{\omega^4}{k^3 {v_{Fe}}^3}\cdot\frac{M}{m}$$

$$\delta^l(k) = -2\pi k v_{Fi}\cdot\frac{{v_{Fi}}^3}{{v_{Fe}}^3}\cdot\frac{M}{m}exp\left[-\frac{2k^2 {v_{Fi}}^2}{3{\omega_{pi}}^2} - 2\right]$$

Transverse waves in degenerate plasma

$$\varepsilon^{tr}(\omega,k) = 1 - \frac{3\omega_{pe}{}^2}{2\omega^2}\left[1 - \left(1 - \frac{\omega^2}{k^2 v_{Fe}{}^2}\right)\cdot\left(1 - \frac{1}{2}\frac{\omega}{k v_{Fe}}\ln\frac{\omega + k v_{Fe}}{\omega - k v_{Fe}}\right)\right]$$

$$\left[k^2 - \frac{\omega^2}{c^2}\varepsilon^{tr}(\omega,k)\right]^2 = 0$$

High phase velocities $\frac{\omega}{k} \gg v_{Fe}$ $\quad x = \frac{k v_{Fe}}{\omega} \ll 1$

$$\varepsilon^{tr}(\omega,k) = 1 - \frac{3\omega_{pe}{}^2}{2\omega^2}\left[1 - \left(1 - \frac{1}{x^2}\right)\cdot\left(1 - \frac{1}{2x}\ln\frac{1+x}{1-x}\right)\right]$$

$$\ln(1+x) \approx x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5}$$

$$\ln\frac{1+x}{1-x} = \ln(1+x) - \ln(1-x) \approx x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \left(x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5}\right) = 2x\cdot\left(1 + \frac{x^2}{3} + \frac{x^4}{5}\right)$$

$$\left(\ldots - \ldots\right) = 1 - \frac{1}{2x}\ln\frac{1+x}{1-x} = -\frac{x^2}{3} - \frac{x^4}{5} = -\frac{x^2}{3}\left(1 + \frac{3x^2}{5}\right) \approx -\frac{x^2}{3}$$

$$\left(\ldots - \ldots\right)\cdot\left(\ldots - \ldots\right) = \left(1 - \frac{1}{x^2}\right)\cdot\left(-\frac{x^2}{3}\right) \approx \frac{1}{3} \quad \Longrightarrow \quad \left[\ldots - \ldots\right] = \left[1 - \frac{1}{3}\right] = \frac{2}{3}$$

$$\varepsilon^{tr}(\omega,k) = 1 - \frac{\omega_{pe}{}^2}{\omega^2}$$

Transverse waves in degenerate plasma

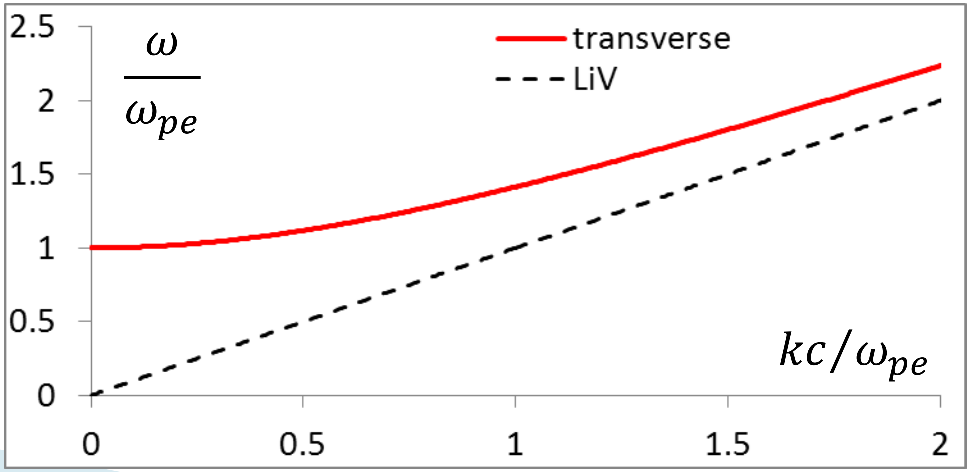
$$\varepsilon^{tr}(\omega,k) = 1 - \frac{3{\omega_{pe}}^2}{2\omega^2}\left[1 - \left(1 - \frac{\omega^2}{k^2 {v_{Fe}}^2}\right)\cdot\left(1 - \frac{1}{2}\frac{\omega}{k v_{Fe}}\ln\frac{\omega + k v_{Fe}}{\omega - k v_{Fe}}\right)\right]$$

$$\left[k^2 - \frac{\omega^2}{c^2}\varepsilon^{tr}(\omega,k)\right]^2 = 0$$

High phase velocities $\frac{\omega}{k} \gg v_{Fe}$ $\quad x = \frac{k v_{Fe}}{\omega} \ll 1$

$$\varepsilon^{tr}(\omega,k) = 1 - \frac{3{\omega_{pe}}^2}{2\omega^2}\left[1 - \left(1 - \frac{1}{x^2}\right)\cdot\left(1 - \frac{1}{2x}\ln\frac{1+x}{1-x}\right)\right] \Rightarrow \varepsilon^{tr}(\omega,k) \approx 1 - \frac{{\omega_{pe}}^2}{\omega^2}$$

$$k^2 = \frac{\omega^2}{c^2}\varepsilon^{tr}(\omega,k) = \frac{\omega^2 - {\omega_{pe}}^2}{c^2} \Rightarrow \omega^2 = {\omega_{pe}}^2 + k^2 c^2$$



Anomalous skin-effect in degenerate plasma

NB: for the nondegenerate plasma

$$\lambda_{sk} = \frac{1}{Im\, k} \approx \left(\sqrt{\frac{2}{\pi}} \cdot \frac{c^2 v_{Te}}{\omega_{pe}{}^2 \omega}\right)^{1/3} \sim \frac{1}{\omega^{1/3}}$$

For the degenerate plasma:

$$\varepsilon^{tr}(\omega, k) = 1 - \frac{3\omega_{pe}{}^2}{2\omega^2}\left[1 - \left(1 - \frac{\omega^2}{k^2 v_{Fe}{}^2}\right) \cdot \left(1 - \frac{1}{2}\frac{\omega}{k v_{Fe}} \ln \frac{\omega + k v_{Fe}}{\omega - k v_{Fe}}\right)\right]$$

Low frequencies: $\omega \ll k v_{Fe}$ $\qquad x = \dfrac{\omega}{k v_{Fe}} \ll 1$

$$\varepsilon^{tr}(\omega, k) = 1 - \frac{3\omega_{pe}{}^2}{2\omega^2}\left[1 - (1 - x^2) \cdot \left(1 - \frac{x}{2} \ln \frac{x+1}{x-1}\right)\right] \approx 1 - \frac{3\omega_{pe}{}^2}{4\omega^2} x \cdot \ln \frac{x+1}{x-1} \approx 1 + i \frac{3\pi\omega_{pe}{}^2}{4\omega k v_{Fe}}$$

$\ln(\omega - k v_{Fe}) = \ln|\omega - k v_{Fe}| - i\pi\delta(\omega - k v_{Fe})$ only if $\omega < k v_{Fe}$!

Anomalous skin-effect in degenerate plasma

NB: for the nondegenerate plasma

$$\lambda_{sk} = \frac{1}{Im\, k} \approx \left(\sqrt{\frac{2}{\pi}} \cdot \frac{c^2 v_{Te}}{{\omega_{pe}}^2 \omega}\right)^{1/3} \sim \frac{1}{\omega^{1/3}}$$

For the degenerate plasma:

$$\varepsilon^{tr}(\omega, k) = 1 - \frac{3{\omega_{pe}}^2}{2\omega^2}\left[1 - \left(1 - \frac{\omega^2}{k^2 {v_{Fe}}^2}\right) \cdot \left(1 - \frac{1}{2}\frac{\omega}{k v_{Fe}} \ln \frac{\omega + k v_{Fe}}{\omega - k v_{Fe}}\right)\right]$$

Low frequencies:

$$\omega \ll k v_{Fe}$$

$$\varepsilon^{tr}(\omega, k) \approx 1 + i\frac{3\pi {\omega_{pe}}^2}{4\omega k v_{Fe}}$$

$$k^2 = \frac{\omega^2}{c^2}\varepsilon^{tr}(\omega, k) \approx i\frac{3\pi {\omega_{pe}}^2 \omega}{4 k v_{Fe} c^2}$$

$$k^3 = i\frac{3\pi {\omega_{pe}}^2 \omega}{4 v_{Fe} c^2} \Longrightarrow Im(k) \approx \frac{1}{2}\left(\frac{3\pi {\omega_{pe}}^2 \omega}{4 v_{Fe} c^2}\right)^{1/3}$$

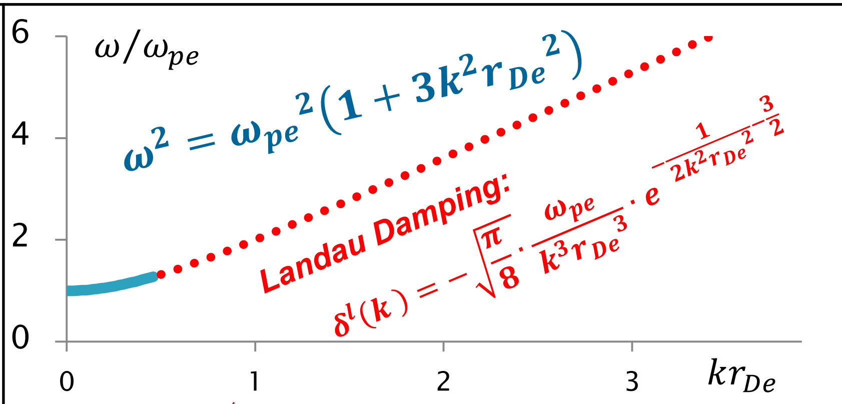
$$\boldsymbol{\lambda_{sk} = \frac{1}{Im\, k} = 2\left(\frac{4 v_{Fe} c^2}{3\pi {\omega_{pe}}^2 \omega}\right)^{1/3} \sim \frac{1}{\omega^{1/3}}}$$

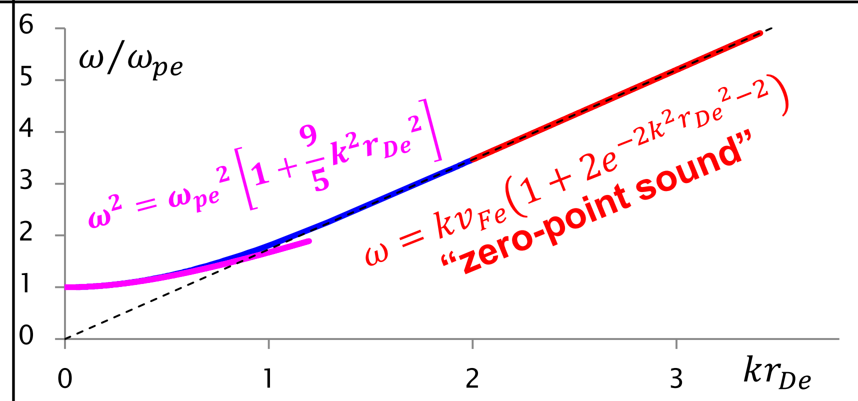
$$\omega \to 0 \Rightarrow \lambda_{sk} \to \infty$$

Waves in colisionless plasma

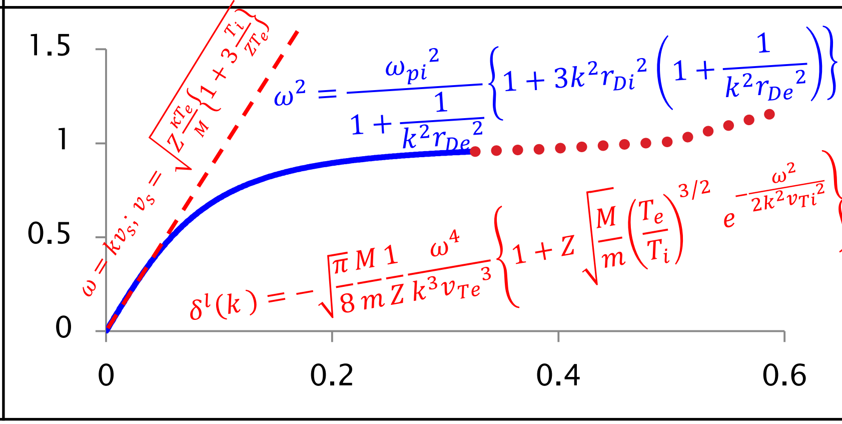
	nondegenerate	degenerate

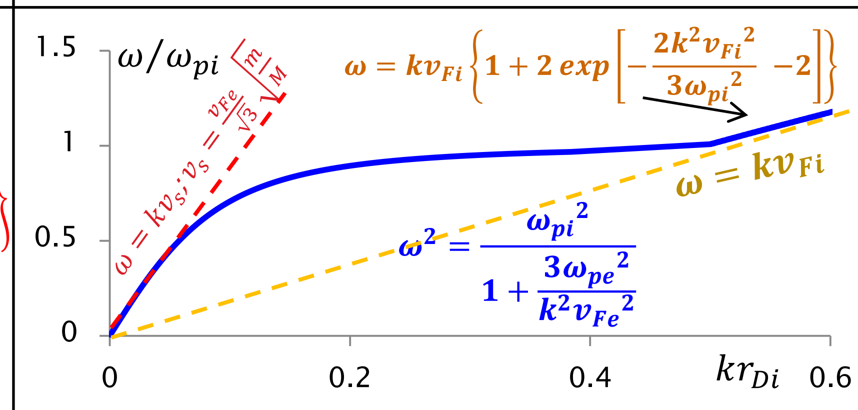
longitudinal — High frequency





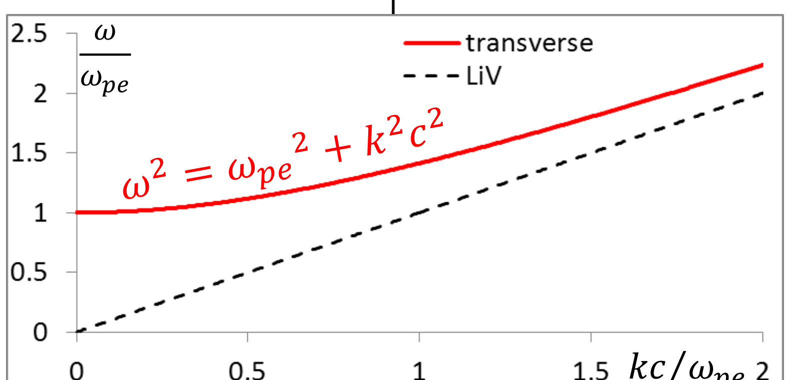
longitudinal — Low frequency





transverse

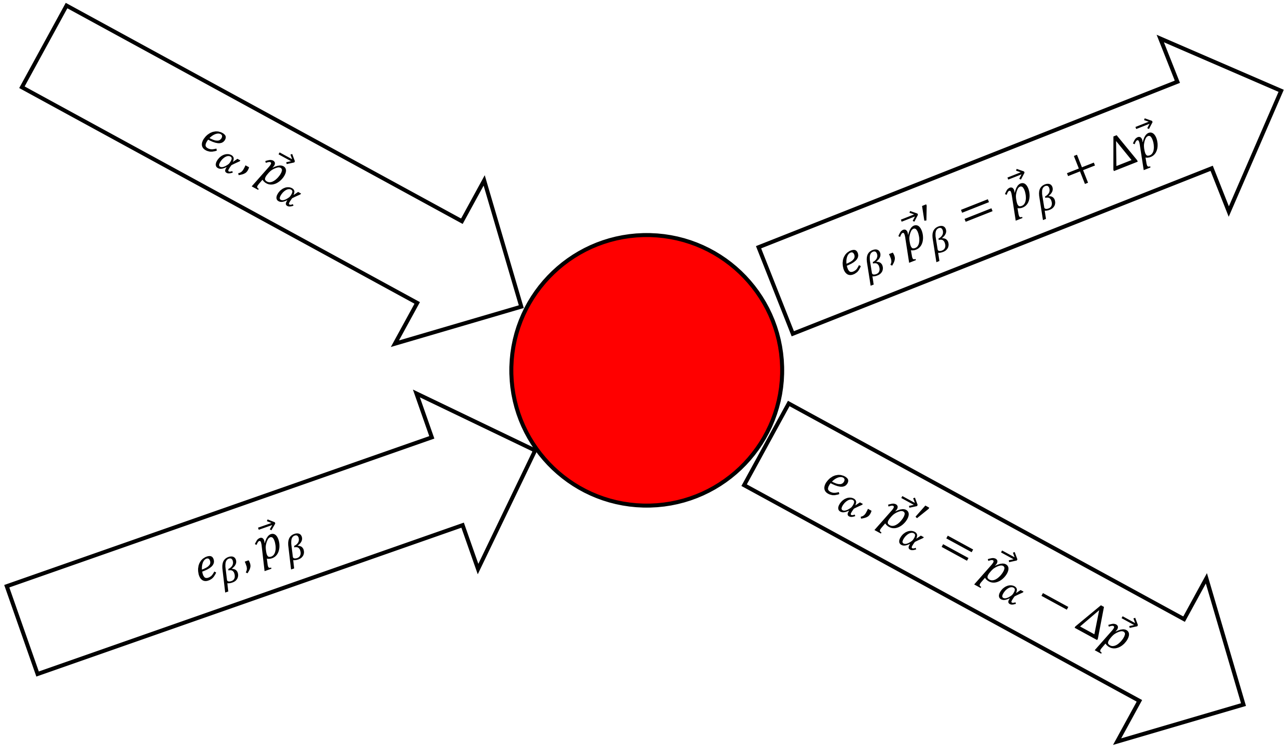
$$\lambda_{sk} = \left(\sqrt{\frac{2}{\pi}} \cdot \frac{c^2 v_{Te}}{\omega_{pe}^2 \omega}\right)^{1/3}$$



$$\lambda_{sk} = 2\left(\frac{4v_{Fe}c^2}{3\pi\omega_{pe}^2\omega}\right)^{1/3}$$

Particle collisions

Vlasov equation w/o collisions: $\frac{\partial f_\alpha}{\partial t} + \vec{v}\frac{\partial f_\alpha}{\partial \vec{r}} + e_\alpha\left(\vec{E} + [\vec{v} \times \vec{B}]\right)\frac{\partial f_\alpha}{\partial \vec{p}} = 0$



$$\frac{\partial f_\alpha}{\partial t} + \vec{v}\frac{\partial f_\alpha}{\partial \vec{r}} + e_\alpha\left(\vec{E} + [\vec{v} \times \vec{B}]\right)\frac{\partial f_\alpha}{\partial \vec{p}} = \left(\frac{\partial f_\alpha}{\partial t}\right)_{col} \Rightarrow collision\ integral$$

Kinetic approach to plasma (L6)

- particle distribution function for **N** particles:

$$f_N(t,\overrightarrow{r_1},\overrightarrow{r_2},\dots,\overrightarrow{r_N},\overrightarrow{p_1},\overrightarrow{p_2},\dots,\overrightarrow{p_N})$$

- for noninteracting particles (**collisionless** plasma)

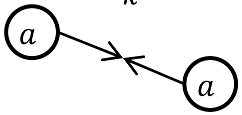
$$f_N(t,\overrightarrow{r_1},\overrightarrow{r_2},\dots,\overrightarrow{r_N},\overrightarrow{p_1},\overrightarrow{p_2},\dots,\overrightarrow{p_N}) = \prod_{i=1}^{N} f(t,\overrightarrow{r_i},\overrightarrow{p_i})$$

- **probability** that the particle is within the volume $d\vec{r}\; d\vec{p}$ around the point $\vec{r}\,,\vec{p}$ of the phase space at the t time moment:

$$f(t,\vec{r},\vec{p})\; d\vec{r}\; d\vec{p}$$

Plasma (gas) parameter:

$$\eta = \frac{U_{pot}}{E_k} \ll 1$$



e.g. for Coulomb interaction:

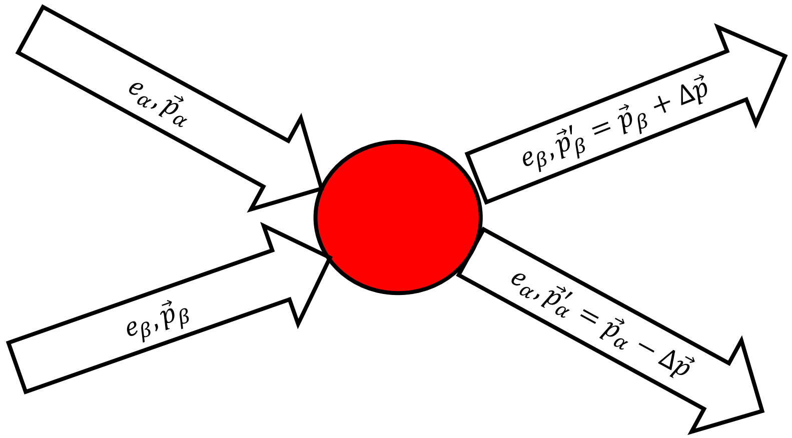
$$\eta = \frac{e^2 N^{1/3}}{4\pi\varepsilon_0 kT} \sim \left(\frac{\langle r\rangle}{r_{De}}\right)^2 \ll 1$$

$\Rightarrow \eta^0 \Rightarrow$ VlasEQN

Kinetic approach to collisions in plasma

$? \eta^1$ ➔ L. Boltzmann

$$\frac{df_\alpha}{dt} = \left(\frac{\partial f_\alpha}{\partial t}\right)_{col} \equiv \sum_\beta \left(\frac{\partial f_\alpha}{\partial t}\right)^{\alpha\beta} \equiv \sum_\beta I_{\alpha\beta}(f_\alpha, f_\beta)$$



Boltzmann collision integral:

$$I_{\alpha\beta}(f_\alpha, f_\beta) = -\int d\boldsymbol{p}_{\boldsymbol{\beta}} d\boldsymbol{p}'_{\boldsymbol{\beta}} d\boldsymbol{p}'_{\boldsymbol{\alpha}}\, W(\boldsymbol{p}_{\boldsymbol{\alpha}}, \boldsymbol{p}_{\boldsymbol{\beta}}, \boldsymbol{p}'_{\boldsymbol{\alpha}}, \boldsymbol{p}'_{\boldsymbol{\beta}}) \cdot [f_\alpha(\boldsymbol{p}_{\boldsymbol{\alpha}}) f_\beta(\boldsymbol{p}_{\boldsymbol{\beta}}) - f_\alpha(\boldsymbol{p}'_{\boldsymbol{\alpha}}) f_\beta(\boldsymbol{p}'_{\boldsymbol{\beta}})]$$

scattering probability

Boltzmann (elastic) collision integral

$$I_{\alpha\beta}(f_\alpha, f_\beta) = -\int d\boldsymbol{p}_\beta d\boldsymbol{p}'_\beta d\boldsymbol{p}'_\alpha \, W(\boldsymbol{p}_\alpha, \boldsymbol{p}_\beta, \boldsymbol{p}'_\alpha, \boldsymbol{p}'_\beta) \cdot [f_\alpha(\boldsymbol{p}_\alpha) f_\beta(\boldsymbol{p}_\beta) - f_\alpha(\boldsymbol{p}'_\alpha) f_\beta(\boldsymbol{p}'_\beta)]$$

Problem: to calculate the probability $W(\boldsymbol{p}_\alpha, \boldsymbol{p}_\beta, \boldsymbol{p}'_\alpha, \boldsymbol{p}'_\beta)$.

Suppose: the potential energy of interactions of two particles is the function of their distance only:

$$U(|\boldsymbol{r}_\alpha - \boldsymbol{r}_\beta|) = \int d\boldsymbol{k} \, U(\boldsymbol{k}) e^{i\boldsymbol{k}\cdot(\boldsymbol{r}_\alpha - \boldsymbol{r}_\beta)}$$

The Fourier transform $U(\boldsymbol{k})$ of the interaction potential $U(\boldsymbol{r})$ is given:

$$U(\boldsymbol{k}) = \frac{1}{(2\pi)^3} \int d\boldsymbol{r} \, U(|\boldsymbol{r}|) e^{-i\boldsymbol{k}\cdot\boldsymbol{r}}$$

How to proceed?

! Let us use the quantum mechanical (QM) approach and Born approximation:

the probability of scattering of 2 particles → from evaluation of the elements of the interaction matrix using unperturbed initial and final states (interaction energy is small compared to the kinetic energy:

(|U| · interaction radius ≪ $\hbar$ · velocity of of the scattered particle)

$$W(\boldsymbol{p}_\alpha, \boldsymbol{p}_\beta, \boldsymbol{p}'_\alpha, \boldsymbol{p}'_\beta) = \frac{2\pi}{\hbar} \left| U_{\boldsymbol{p}_\alpha, \boldsymbol{p}_\beta, \boldsymbol{p}'_\alpha, \boldsymbol{p}'_\beta} \right|^2 \delta[(\mathcal{E}'_\alpha + \mathcal{E}'_\beta) - (\mathcal{E}_\alpha + \mathcal{E}_\beta)]$$

Summary

- Longitudinal low frequency waves in degenerate plasma:
  - Intermediate phase velocity range $v_{Fi} \ll \frac{\omega}{k} \ll v_{Fe}$

$$\omega^2 = {\omega_{pi}}^2 / \left(1 + \frac{3{\omega_{pe}}^2}{k^2 {v_{Fe}}^2}\right); \qquad \delta^l(k) = -\frac{3\pi}{4}\frac{\omega^4}{k^3 {v_{Fe}}^3} \cdot \frac{M}{m}$$

  - Ion-acoustic waves: $\omega = kv_s;\ \ v_s = \frac{v_{Fe}}{\sqrt{3}}\sqrt{\frac{m}{M}}$
  - Damping in partially degenerate plasma: $\delta^l(k) = -\frac{3\pi}{4}\frac{\omega^4}{k^3 {v_{Fe}}^3} \cdot \frac{M}{m} - \sqrt{\frac{\pi}{8}}\frac{\omega^4}{k^3 {v_{Ti}}^3} e^{-\frac{\omega^2}{2k^2 {v_{Ti}}^2}}$
- Transverse waves in degenerate plasma, skin-effect
  - High frequency $\omega^2 = {\omega_{pe}}^2 + k^2 c^2$
  - Anomalous skin-effect: $\lambda_{sk} = 2\left(\frac{4 v_{Fe} c^2}{3\pi {\omega_{pe}}^2 \omega}\right)^{1/3} \sim \frac{1}{\omega^{1/3}}$
- General classification of waves in collisionless plasma
  - Transverse (high and low frequency) and longitudinal
  - Degenerate and nondegenerate plasma
- Kinetic approach with particle collisions
  - Boltzmann collision integral - intro
    - Next:
      - Collision integrals → Fokker-Planck EQN:
        - Charged particles → Lenard-Balescu collision integral, Landau formula, Coulomb logarithm
        - Model integral for elastic particle scattering – BGK
        - ?Degenerate plasma