Kinetic equation
with (charged) particle collisions

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"Electrodynamics of plasma and beams“ – 2012/2013
L#10

Discussed last time…

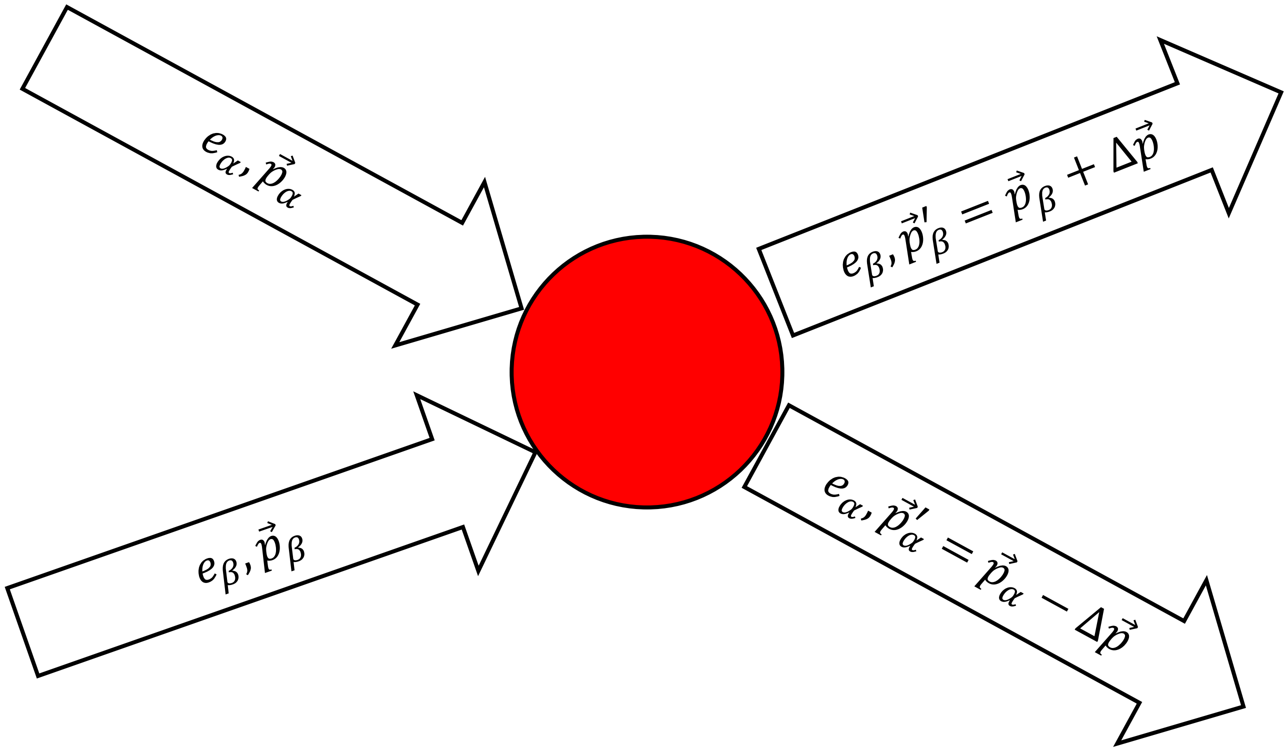
- Longitudinal low frequency waves in degenerate plasma:
  - Intermediate phase velocity range $v_{Fi} \ll \frac{\omega}{k} \ll v_{Fe}$

$$\omega^2 = {\omega_{pi}}^2 \Big/ \left(1 + \frac{3{\omega_{pe}}^2}{k^2 {v_{Fe}}^2}\right); \qquad \delta^l(k) = -\frac{3\pi}{4} \frac{\omega^4}{k^3 {v_{Fe}}^3} \cdot \frac{M}{m}$$

  - Ion-acoustic waves: $\omega = k v_s;\ \ v_s = \frac{v_{Fe}}{\sqrt{3}} \sqrt{\frac{m}{M}}$
  - Damping in partially degenerate plasma: $\delta^l(k) = -\frac{3\pi}{4} \frac{\omega^4}{k^3 {v_{Fe}}^3} \cdot \frac{M}{m} - \sqrt{\frac{\pi}{8}} \frac{\omega^4}{k^3 {v_{Ti}}^3} e^{-\frac{\omega^2}{2k^2 {v_{Ti}}^2}}$
- Transverse waves in degenerate plasma, skin-effect
  - High frequency $\omega^2 = {\omega_{pe}}^2 + k^2 c^2$
  - Anomalous skin-effect: $\lambda_{sk} = 2\left(\frac{4 v_{Fe} c^2}{3\pi {\omega_{pe}}^2 \omega}\right)^{1/3} \sim \frac{1}{\omega^{1/3}}$
- General classification of waves in collisionless plasma
  - Transverse (high and low frequency) and longitudinal
  - Degenerate and nondegenerate plasma
- Kinetic approach with particle collisions
  - Boltzmann collision integral - intro
    - Not yet:
      - Collision integrals → Fokker-Planck EQN:
        - Charged particles → Lenard-Balescu collision integral, Landau formula, Coulomb logarithm
        - Model integral for elastic particle scattering – BGK
        - ?Degenerate plasma

Particle collisions

Vlasov equation w/o collisions: $\frac{\partial f_\alpha}{\partial t} + \vec{v}\frac{\partial f_\alpha}{\partial \vec{r}} + e_\alpha\left(\vec{E} + [\vec{v} \times \vec{B}]\right)\frac{\partial f_\alpha}{\partial \vec{p}} = 0$



$$\frac{\partial f_\alpha}{\partial t} + \vec{v}\frac{\partial f_\alpha}{\partial \vec{r}} + e_\alpha\left(\vec{E} + [\vec{v} \times \vec{B}]\right)\frac{\partial f_\alpha}{\partial \vec{p}} = \left(\frac{\partial f_\alpha}{\partial t}\right)_{col} \Rightarrow \textit{collision integral}$$

Kinetic approach to plasma (L6)

- particle distribution function for **N** particles:

$$f_N(t,\overrightarrow{r_1},\overrightarrow{r_2},\dots,\overrightarrow{r_N},\overrightarrow{p_1},\overrightarrow{p_2},\dots,\overrightarrow{p_N})$$

- for noninteracting particles (**collisionless** plasma)

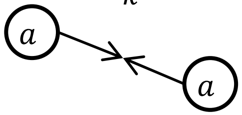
$$f_N(t,\overrightarrow{r_1},\overrightarrow{r_2},\dots,\overrightarrow{r_N},\overrightarrow{p_1},\overrightarrow{p_2},\dots,\overrightarrow{p_N}) = \prod_{i=1}^{N} f(t,\overrightarrow{r_i},\overrightarrow{p_i})$$

- **probability** that the particle is within the volume $d\vec{r}\,d\vec{p}$ around the point $\vec{r}\,,\vec{p}$ of the phase space at the t time moment:

$$f(t,\vec{r},\vec{p})\,d\vec{r}\,d\vec{p}$$

Plasma (gas) parameter:

$$\eta = \frac{U_{pot}}{E_k} \ll 1$$



e.g. for Coulomb interaction:

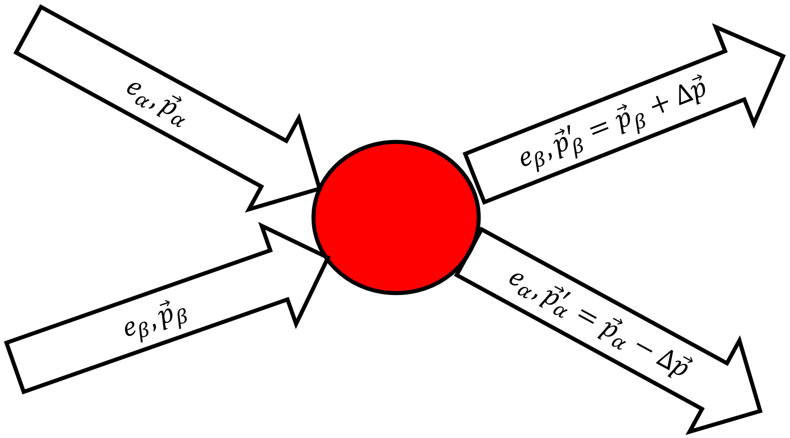
$$\eta = \frac{e^2 N^{1/3}}{4\pi\varepsilon_0 kT} \sim \left(\frac{\langle r\rangle}{r_{De}}\right)^2 \ll 1$$

$\Longrightarrow \eta^0 \Longrightarrow$ VlasEQN

Kinetic approach to collisions in plasma

$?\eta^1$ ➔ L. Boltzmann

$$\frac{df_\alpha}{dt} = \left(\frac{\partial f_\alpha}{\partial t}\right)_{col} \equiv \sum_\beta \left(\frac{\partial f_\alpha}{\partial t}\right)^{\alpha\beta} \equiv \sum_\beta I_{\alpha\beta}(f_\alpha, f_\beta)$$



Boltzmann collision integral:

$$I_{\alpha\beta}(f_\alpha, f_\beta) = -\int d\boldsymbol{p}_\beta d\boldsymbol{p}'_\beta d\boldsymbol{p}'_\alpha \, W(\boldsymbol{p}_\alpha, \boldsymbol{p}_\beta, \boldsymbol{p}'_\alpha, \boldsymbol{p}'_\beta) \cdot [f_\alpha(\boldsymbol{p}_\alpha) f_\beta(\boldsymbol{p}_\beta) - f_\alpha(\boldsymbol{p}'_\alpha) f_\beta(\boldsymbol{p}'_\beta)]$$

scattering probability

Boltzmann (elastic) collision integral

$$I_{\alpha\beta}(f_\alpha, f_\beta) = -\int d\boldsymbol{p}_{\boldsymbol{\beta}} d\boldsymbol{p}'_{\boldsymbol{\beta}} d\boldsymbol{p}'_{\boldsymbol{\alpha}}\, W(\boldsymbol{p}_{\boldsymbol{\alpha}}, \boldsymbol{p}_{\boldsymbol{\beta}}, \boldsymbol{p}'_{\boldsymbol{\alpha}}, \boldsymbol{p}'_{\boldsymbol{\beta}}) \cdot [f_\alpha(\boldsymbol{p}_{\boldsymbol{\alpha}}) f_\beta(\boldsymbol{p}_{\boldsymbol{\beta}}) - f_\alpha(\boldsymbol{p}'_{\boldsymbol{\alpha}}) f_\beta(\boldsymbol{p}'_{\boldsymbol{\beta}})]$$

Problem: to calculate the probability $W(\boldsymbol{p}_{\boldsymbol{\alpha}}, \boldsymbol{p}_{\boldsymbol{\beta}}, \boldsymbol{p}'_{\boldsymbol{\alpha}}, \boldsymbol{p}'_{\boldsymbol{\beta}})$.

Suppose: the potential energy of interactions of two particles is the function of their distance only:

$$U(|\boldsymbol{r}_{\boldsymbol{\alpha}} - \boldsymbol{r}_\beta|) = \int d\boldsymbol{k}\, U(\boldsymbol{k}) e^{i\boldsymbol{k}\cdot(\boldsymbol{r}_{\boldsymbol{\alpha}} - \boldsymbol{r}_\beta)}$$

The Fourier transform $U(\boldsymbol{k})$ of the interaction potential $U(\boldsymbol{r})$ is given:

$$U(\boldsymbol{k}) = \frac{1}{(2\pi)^3} \int d\boldsymbol{r}\, U(|\boldsymbol{r}|) e^{-i\boldsymbol{k}\cdot\boldsymbol{r}}$$

How to proceed?

! Let us use the quantum mechanical (QM) approach and Born approximation:

the probability of scattering of 2 particles → from evaluation of the elements of the interaction matrix using unperturbed initial and final states (interaction energy is small compared to the kinetic energy:

(|U| · interaction radius ≪ $\hbar$ · velocity of of the scattered particle)

$$W(\boldsymbol{p}_{\boldsymbol{\alpha}}, \boldsymbol{p}_{\boldsymbol{\beta}}, \boldsymbol{p}'_{\boldsymbol{\alpha}}, \boldsymbol{p}'_{\boldsymbol{\beta}}) = \frac{2\pi}{\hbar} \left| U_{\boldsymbol{p}_{\boldsymbol{\alpha}}, \boldsymbol{p}_{\boldsymbol{\beta}}, \boldsymbol{p}'_{\boldsymbol{\alpha}}, \boldsymbol{p}'_{\boldsymbol{\beta}}} \right|^2 \delta[(\mathcal{E}'_\alpha + \mathcal{E}'_\beta) - (\mathcal{E}_{\boldsymbol{\alpha}} + \mathcal{E}_\beta)]$$

Born approximation

$$I_{\alpha\beta}(f_\alpha, f_\beta) = -\int d\boldsymbol{p}_\beta d\boldsymbol{p}'_\beta d\boldsymbol{p}'_\alpha \, W(\boldsymbol{p}_\alpha, \boldsymbol{p}_\beta, \boldsymbol{p}'_\alpha, \boldsymbol{p}'_\beta) \cdot [f_\alpha(\boldsymbol{p}_\alpha) f_\beta(\boldsymbol{p}_\beta) - f_\alpha(\boldsymbol{p}'_\alpha) f_\beta(\boldsymbol{p}'_\beta)]$$

$$W(\boldsymbol{p}_\alpha, \boldsymbol{p}_\beta, \boldsymbol{p}'_\alpha, \boldsymbol{p}'_\beta) = \frac{2\pi}{\hbar} \left| U_{\boldsymbol{p}_\alpha, \boldsymbol{p}_\beta, \boldsymbol{p}'_\alpha, \boldsymbol{p}'_\beta} \right|^2 \delta[(\mathcal{E}'_\alpha + \mathcal{E}'_\beta) - (\mathcal{E}_\alpha + \mathcal{E}_\beta)]$$

The wave function of a free nonrelativistic particle with momentum $\boldsymbol{p}$ and energy $\mathcal{E} = \frac{\boldsymbol{p}^2}{2m}$:

$$\Psi_p = \frac{1}{(2\pi\hbar)^3} \exp\left(-i\frac{\mathcal{E}}{\hbar}t + i\frac{\boldsymbol{p}\cdot\boldsymbol{r}}{\hbar}\right)$$

The interaction matrix:

$$U_{\boldsymbol{p}_\alpha, \boldsymbol{p}_\beta, \boldsymbol{p}'_\alpha, \boldsymbol{p}'_\beta} = \int d\boldsymbol{k} \, U(\boldsymbol{k}) \langle \boldsymbol{p}'_\alpha | e^{i\boldsymbol{k}\cdot\boldsymbol{r}_\alpha} | \boldsymbol{p}_\alpha \rangle \langle \boldsymbol{p}'_\beta | e^{-i\boldsymbol{k}\cdot\boldsymbol{r}_\beta} | \boldsymbol{p}_\beta \rangle$$

Born approximation

$$W(\boldsymbol{p}_\alpha, \boldsymbol{p}_\beta, \boldsymbol{p}'_\alpha, \boldsymbol{p}'_\beta) = \frac{2\pi}{\hbar}\left|U_{\boldsymbol{p}_\alpha \boldsymbol{p}_\beta \boldsymbol{p}'_\alpha \boldsymbol{p}'_\beta}\right|^2 \delta\left[(\mathcal{E}'_\alpha + \mathcal{E}'_\beta) - (\mathcal{E}_\alpha + \mathcal{E}_\beta)\right]$$

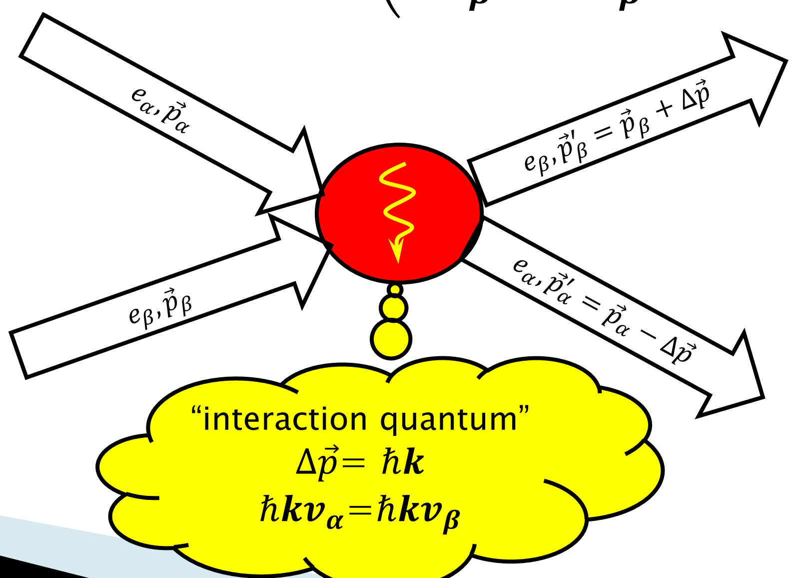
The interaction matrix: $U_{\boldsymbol{p}_\alpha \boldsymbol{p}_\beta \boldsymbol{p}'_\alpha \boldsymbol{p}'_\beta} = \int d\boldsymbol{k}\, U(\boldsymbol{k}) \langle \boldsymbol{p}'_\alpha | e^{i\boldsymbol{k}\cdot\boldsymbol{r}_\alpha} | \boldsymbol{p}_\alpha \rangle \langle \boldsymbol{p}'_\beta | e^{i\boldsymbol{k}\cdot\boldsymbol{r}_\beta} | \boldsymbol{p}_\beta \rangle$

$$\boldsymbol{r}_\alpha = \boldsymbol{r}_{\alpha 0} + \boldsymbol{v}_\alpha \boldsymbol{t}$$

$$\Psi_p = \frac{1}{(2\pi\hbar)^3} \exp\left(-i\frac{\mathcal{E}}{\hbar}t + i\frac{\boldsymbol{p}\cdot\boldsymbol{r}}{\hbar}\right)$$

$$\langle \boldsymbol{p}'_\alpha | e^{i\boldsymbol{k}\cdot\boldsymbol{r}_\alpha} | \boldsymbol{p}_\alpha \rangle = \delta(\boldsymbol{p}'_\alpha - \boldsymbol{p}_\alpha + \hbar\boldsymbol{k})\,\delta\left(\frac{{\boldsymbol{p}'_\alpha}^2}{2m_\alpha} - \frac{{\boldsymbol{p}_\alpha}^2}{2m_\alpha} + \hbar\boldsymbol{k}\boldsymbol{v}_\alpha\right)$$

$$\langle \boldsymbol{p}'_\beta | e^{-i\boldsymbol{k}\cdot\boldsymbol{r}_\beta} | \boldsymbol{p}_\beta \rangle = \delta(\boldsymbol{p}'_\beta - \boldsymbol{p}_\beta - \hbar\boldsymbol{k})\,\delta\left(\frac{{\boldsymbol{p}'_\beta}^2}{2m_\beta} - \frac{{\boldsymbol{p}_\beta}^2}{2m_\beta} - \hbar\boldsymbol{k}\boldsymbol{v}_\beta\right)$$



Boltzmann (elastic) collision integral

$$\left(\frac{\partial f_\alpha}{\partial t}\right)^{\alpha\beta} = I_{\alpha\beta}(f_\alpha, f_\beta) = -\int d\boldsymbol{p}_\beta d\boldsymbol{p}'_\beta d\boldsymbol{p}'_\alpha \, W(\boldsymbol{p}_\alpha, \boldsymbol{p}_\beta, \boldsymbol{p}'_\alpha, \boldsymbol{p}'_\beta) \cdot [f_\alpha(\boldsymbol{p}_\alpha) f_\beta(\boldsymbol{p}_\beta) - f_\alpha(\boldsymbol{p}'_\alpha) f_\beta(\boldsymbol{p}'_\beta)]$$

$$W(\boldsymbol{p}_\alpha, \boldsymbol{p}_\beta, \boldsymbol{p}'_\alpha, \boldsymbol{p}'_\beta) = \frac{2\pi}{\hbar} \left| U_{\boldsymbol{p}_\alpha, \boldsymbol{p}_\beta, \boldsymbol{p}'_\alpha, \boldsymbol{p}'_\beta} \right|^2 \delta[(\mathcal{E}'_\alpha + \mathcal{E}'_\beta) - (\mathcal{E}_\alpha + \mathcal{E}_\beta)]$$

$$U_{\boldsymbol{p}_\alpha, \boldsymbol{p}_\beta, \boldsymbol{p}'_\alpha, \boldsymbol{p}'_\beta} = \int d\boldsymbol{k} \, U(\boldsymbol{k}) \langle \boldsymbol{p}'_\alpha | \boldsymbol{e}^{i\boldsymbol{k}\cdot\boldsymbol{r}_\alpha} | \boldsymbol{p}_\alpha \rangle \langle \boldsymbol{p}'_\beta | \boldsymbol{e}^{i\boldsymbol{k}\cdot\boldsymbol{r}_\beta} | \boldsymbol{p}_\beta \rangle$$

$$\langle \boldsymbol{p}'_\alpha | \boldsymbol{e}^{i\boldsymbol{k}\cdot\boldsymbol{r}_\alpha} | \boldsymbol{p}_\alpha \rangle = \delta(\boldsymbol{p}'_\alpha - \boldsymbol{p}_\alpha + \hbar\boldsymbol{k}) \delta\left(\frac{{\boldsymbol{p}'_\alpha}^2}{2m_\alpha} - \frac{{\boldsymbol{p}_\alpha}^2}{2m_\alpha} + \hbar\boldsymbol{k}\boldsymbol{v}_\alpha\right)$$

$$\langle \boldsymbol{p}'_\beta | \boldsymbol{e}^{-i\boldsymbol{k}\cdot\boldsymbol{r}_\beta} | \boldsymbol{p}_\beta \rangle = \delta(\boldsymbol{p}'_\beta - \boldsymbol{p}_\beta - \hbar\boldsymbol{k}) \delta\left(\frac{{\boldsymbol{p}'_\beta}^2}{2m_\beta} - \frac{{\boldsymbol{p}_\beta}^2}{2m_\beta} - \hbar\boldsymbol{k}\boldsymbol{v}_\beta\right)$$

$$I_{\alpha\beta}(f_\alpha, f_\beta) = \frac{2\pi}{\hbar} \int d\boldsymbol{p}_\beta \frac{d\boldsymbol{k}}{(2\pi)^3} |U(\boldsymbol{k})|^2 \cdot \delta\left(\frac{(\boldsymbol{p}_\alpha - \hbar\boldsymbol{k})^2}{2m_\alpha} + \frac{(\boldsymbol{p}_\beta + \hbar\boldsymbol{k})^2}{2m_\beta} - \frac{{\boldsymbol{p}_\alpha}^2}{2m_\alpha} - \frac{{\boldsymbol{p}_\beta}^2}{2m_\beta}\right) \times$$
$$\times [f_\alpha(\boldsymbol{p}_\alpha) f_\beta(\boldsymbol{p}_\beta) - f_\alpha(\boldsymbol{p}_\alpha - \hbar\boldsymbol{k}) f_\beta(\boldsymbol{p}_\beta + \hbar\boldsymbol{k})]$$

Classical limit: $\hbar\boldsymbol{k} \ll \boldsymbol{p}_\alpha, \boldsymbol{p}_\beta$

Boltzmann (elastic) collision integral

Classical limit: $\hbar \boldsymbol{k} \ll \boldsymbol{p_\alpha}, \boldsymbol{p_\beta}$

$\boldsymbol{u} = \boldsymbol{v_\alpha} - \boldsymbol{v_\beta}$

$$I_{\alpha\beta}(f_\alpha, f_\beta) = \frac{2\pi}{\hbar} \int d\boldsymbol{p_\beta} \frac{d\boldsymbol{k}}{(2\pi)^3} |U(\boldsymbol{k})|^2 \cdot \delta\left(\frac{(\boldsymbol{p_\alpha} - \hbar\boldsymbol{k})^2}{2m_\alpha} + \frac{(\boldsymbol{p}_\beta + \hbar\boldsymbol{k})^2}{2m_\beta} - \frac{\boldsymbol{p_\alpha}^2}{2m_\alpha} - \frac{\boldsymbol{p}_\beta^2}{2m_\beta}\right) \times$$

$$\times \left[f_\alpha(\boldsymbol{p_\alpha}) f_\beta(\boldsymbol{p_\beta}) - f_\alpha(\boldsymbol{p_\alpha} - \hbar\boldsymbol{k}) f_\beta(\boldsymbol{p}_\beta + \hbar\boldsymbol{k})\right]$$

$$\delta(-\hbar\boldsymbol{k}\boldsymbol{u}) = \frac{1}{\hbar}\delta(\boldsymbol{k}\boldsymbol{u})$$

$$\hbar k_i \cdot [\dots]_i + \frac{\hbar^2 k_i k_i}{2}[\dots]_{ij}$$

$$\hbar k_i \cdot [\dots]_i \rightarrow 0$$

$$I_{\alpha\beta}(f_\alpha, f_\beta) = \frac{\partial}{\partial p_{\alpha i}} \int d\boldsymbol{p_\beta}\; \varkappa_{ij}{}^{\alpha\beta}(\boldsymbol{p_\alpha}, \boldsymbol{p_\beta}) \left[\frac{\partial f_\alpha(\boldsymbol{p_\alpha})}{\partial p_{\alpha j}} f_\beta(\boldsymbol{p_\beta}) - f_\alpha(\boldsymbol{p_\alpha}) \frac{\partial f_\beta(\boldsymbol{p}_\beta)}{\partial p_{\beta j}}\right]$$

$$\varkappa_{ij}{}^{\alpha\beta}(\boldsymbol{p_\alpha}, \boldsymbol{p_\beta}) = \frac{\pi}{2} \cdot \frac{u^2 - u_i u_j}{u^2} \int (2\pi)^3 k^2\, d\boldsymbol{k} \cdot |U(\boldsymbol{k})|^2 \cdot \delta(\boldsymbol{k}\boldsymbol{u})$$

The Fokker-Planck Equation

$$\frac{\partial f_\alpha}{\partial t} + \vec{v}\frac{\partial f_\alpha}{\partial \vec{r}} + e_\alpha\left(\vec{E} + [\vec{v} \times \vec{B}]\right)\frac{\partial f_\alpha}{\partial \vec{p}} = \left(\frac{\partial f_\alpha}{\partial t}\right)_{col} = \sum_\beta \left(\frac{\partial f_\alpha}{\partial t}\right)^{\alpha\beta} \equiv \sum_\beta I_{\alpha\beta}(f_\alpha, f_\beta)$$

$$\frac{\partial f_\alpha}{\partial t} + \vec{v}\frac{\partial f_\alpha}{\partial \vec{r}} + e_\alpha\left(\vec{E} + [\vec{v} \times \vec{B}]\right)\frac{\partial f_\alpha}{\partial \vec{p}} = \frac{\partial}{\partial p_{\alpha i}}\left(D_{ij}\frac{\partial f_\alpha}{\partial p_{\alpha j}} - A_i \cdot f_\alpha\right)$$

Diffusion in momentum space

$$D_{ij} = \sum_\beta \int d\boldsymbol{p_\beta}\ \varkappa_{ij}{}^{\alpha\beta}(\boldsymbol{p_\alpha}, \boldsymbol{p_\beta}) f_\beta(\boldsymbol{p_\beta})$$

Friction in momentum space

$$A_i = \sum_\beta \int d\boldsymbol{p_\beta}\ \varkappa_{ij}{}^{\alpha\beta}(\boldsymbol{p_\alpha}, \boldsymbol{p_\beta}) \frac{\partial f_\beta(\boldsymbol{p_\beta})}{\partial p_{\beta j}}$$

$$\varkappa_{ij}{}^{\alpha\beta}(\boldsymbol{p_\alpha}, \boldsymbol{p_\beta}) = \frac{\pi}{2} \cdot \frac{u^2 - u_i u_j}{u^2} \int (2\pi)^3 k^2\, d\boldsymbol{k} \cdot |U(\boldsymbol{k})|^2 \cdot \delta(\boldsymbol{ku})$$

$??\, U(\boldsymbol{r}) \rightarrow\ U(\boldsymbol{k})$

Collision Integral: charged particles

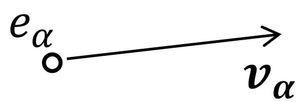
$$\frac{\partial f_\alpha}{\partial t} + \vec{v}\frac{\partial f_\alpha}{\partial \vec{r}} + e_\alpha\left(\vec{E} + [\vec{v}\times\vec{B}]\right)\frac{\partial f_\alpha}{\partial \vec{p}} = \frac{\partial}{\partial p_{\alpha i}}\left(D_{ij}\frac{\partial f_\alpha}{\partial p_{\alpha j}} - A_i \cdot f_\alpha\right)$$

$$D_{ij} = \sum_\beta \int d\boldsymbol{p_\beta}\ \varkappa_{ij}{}^{\alpha\beta}(\boldsymbol{p_\alpha}, \boldsymbol{p_\beta}) f_\beta(\boldsymbol{p_\beta})$$

$$A_i = \sum_\beta \int d\boldsymbol{p_\beta}\ \varkappa_{ij}{}^{\alpha\beta}(\boldsymbol{p_\alpha}, \boldsymbol{p_\beta}) \frac{\partial f_\beta(\boldsymbol{p_\beta})}{\partial p_{\beta j}}$$

$$\varkappa_{ij}{}^{\alpha\beta}(\boldsymbol{p_\alpha}, \boldsymbol{p_\beta}) = \frac{\pi}{2}\cdot\frac{u^2 - u_i u_j}{u^2}\int (2\pi)^3 k^2\, d\boldsymbol{k}\cdot|U(\boldsymbol{k})|^2\cdot\delta(\boldsymbol{ku})$$

$$??\,U(\boldsymbol{r}) \to U(\boldsymbol{k})$$



$$\boldsymbol{r_\alpha} = \boldsymbol{r_{\alpha 0}} + \boldsymbol{v_\alpha} t$$

$$div\ \boldsymbol{D} = e_\alpha\, \delta(\boldsymbol{r} - \boldsymbol{r_{\alpha 0}} - \boldsymbol{v_\alpha} t\ )$$

Collision Integral: charged particles

$$div\,\mathbf{D} = e_\alpha\,\delta(\mathbf{r} - \mathbf{r}_{\alpha 0} - \mathbf{v}_\alpha t\ )$$

From L3

$$\mathbf{D}(t,\vec{r}) = \int_{-\infty}^{\infty} d\omega \int d\vec{\mathbf{k}}\,\mathbf{D}(\omega,\vec{k})e^{-i\omega t + i\vec{k}\vec{r}}$$

$$\mathbf{E}(t,\vec{r}) = \int_{-\infty}^{\infty} d\omega \int d\vec{\mathbf{k}}\,\mathbf{E}(\omega,\vec{k})e^{-i\omega t + i\vec{k}\vec{r}}$$

$$D_i(\omega,\vec{k}) = \varepsilon_0\varepsilon_{ij}(\omega,\vec{k})E_j(\omega,\vec{k})$$

$$\delta(\mathbf{r} - \mathbf{r}_\alpha\ ) = \frac{1}{(2\pi)^3}\int d\mathbf{k}\ e^{i\mathbf{k}(\mathbf{r}-\mathbf{r}_\alpha)}$$

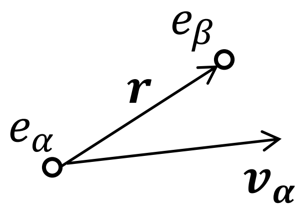
From L4 (Problem 4.1)

$$\mathbf{E}(t,\vec{r}) = -\nabla\Phi(|\mathbf{r}_\alpha - \mathbf{r}|)$$

$$\Phi(|\mathbf{r}_\alpha - \mathbf{r}|) = \frac{e_\alpha}{(2\pi)^3\varepsilon_0}\int d\mathbf{k}\ \frac{e^{i\mathbf{k}(\mathbf{r}_\alpha - \mathbf{r})}}{k_i k_j \varepsilon_{ij}(\mathbf{k}\mathbf{v}_\alpha, \mathbf{k})}$$

Collision Integral: charged particles

$$\Phi(|\boldsymbol{r}_{\boldsymbol{\alpha}} - \boldsymbol{r}|) = \frac{e_\alpha}{(2\pi)^3\varepsilon_0}\int d\boldsymbol{k}\,\frac{e^{i\boldsymbol{k}(\boldsymbol{r}_\alpha - \boldsymbol{r})}}{k_i k_j \varepsilon_{ij}(\boldsymbol{k}\boldsymbol{v}_\alpha, \boldsymbol{k})}$$



$$U(\boldsymbol{r}) = e_\beta\Phi = \frac{e_\alpha e_\beta}{(2\pi)^3\varepsilon_0}\int d\boldsymbol{k}\,\frac{e^{i\boldsymbol{k}\boldsymbol{r}}}{k_i k_j \varepsilon_{ij}(\boldsymbol{k}\boldsymbol{v}_\alpha, \boldsymbol{k})}$$

FT

$$U(\boldsymbol{k}) = \frac{e_\alpha e_\beta}{(2\pi)^3\varepsilon_0}\cdot\frac{1}{k_i k_j \varepsilon_{ij}(\boldsymbol{k}\boldsymbol{v}_\alpha, \boldsymbol{k})}$$

Lenard-Balescu collision Integral

$$U(\boldsymbol{k}) = \frac{e_\alpha e_\beta}{(2\pi)^3 \varepsilon_0} \cdot \frac{1}{k_i k_j \varepsilon_{ij}(\boldsymbol{k v_\alpha}, \boldsymbol{k})}$$

$$\varkappa_{ij}{}^{\alpha\beta}(\boldsymbol{p_\alpha}, \boldsymbol{p_\beta}) = \frac{\pi}{2} \cdot \frac{u^2 - u_i u_j}{u^2} \int (2\pi)^3 k^2 \, d\boldsymbol{k} \cdot |U(\boldsymbol{k})|^2 \cdot \delta(\boldsymbol{ku})$$

$$\varkappa_{ij}{}^{\alpha\beta} = \pi \left(\frac{e_\alpha e_\beta}{8\pi^3 \varepsilon_0} \right)^2 \int (2\pi)^3 \, d\boldsymbol{k} \cdot \frac{k_i k_j \delta(\boldsymbol{k v_\alpha} - \boldsymbol{k v_\beta})}{\left| k_i k_j \varepsilon_{ij}(\boldsymbol{k v_\alpha}, \boldsymbol{k}) \right|^2}$$

Dynamically screened kernel of the collision integral

$$\frac{\partial f_\alpha}{\partial t} + \vec{v} \frac{\partial f_\alpha}{\partial \vec{r}} + e_\alpha \left(\vec{E} + [\vec{v} \times \vec{B}] \right) \frac{\partial f_\alpha}{\partial \vec{p}} = \frac{\partial}{\partial p_{\alpha i}} \left(D_{ij} \frac{\partial f_\alpha}{\partial p_{\alpha j}} - A_i \cdot f_\alpha \right)$$

$$D_{ij} = \sum_\beta \int d\boldsymbol{p_\beta} \; \varkappa_{ij}{}^{\alpha\beta}(\boldsymbol{p_\alpha}, \boldsymbol{p_\beta}) f_\beta(\boldsymbol{p_\beta})$$

$$A_i = \sum_\beta \int d\boldsymbol{p_\beta} \; \varkappa_{ij}{}^{\alpha\beta}(\boldsymbol{p_\alpha}, \boldsymbol{p_\beta}) \frac{\partial f_\beta(\boldsymbol{p}_\beta)}{\partial p_{\beta j}}$$

Landau Formula, Coulomb logarithm

$u = v_\alpha - v_\beta$

$$\varkappa_{ij}{}^{\alpha\beta} = \pi \left(\frac{e_\alpha e_\beta}{8\pi^3 \varepsilon_0} \right)^2 \int (2\pi)^3 \, d\boldsymbol{k} \cdot \frac{k_i k_j \delta(\boldsymbol{k}\boldsymbol{v}_\alpha - \boldsymbol{k}\boldsymbol{v}_\beta)}{\left| k_i k_j \varepsilon_{ij}(\boldsymbol{k}\boldsymbol{v}_\alpha, \boldsymbol{k}) \right|^2}$$

Case of vacuum: $\varepsilon_{ij} = \delta_{ij}$

$$k_{max} \sim \frac{1}{r_{min}}$$

$$\varkappa_{ij}{}^{\alpha\beta} = \frac{1}{8\pi} \left(\frac{e_\alpha e_\beta}{\varepsilon_0} \right)^2 \cdot \frac{u^2 - u_i u_j}{u^2} \cdot \int_0^\infty \frac{dk}{k}$$

$$k_{min} \sim \frac{1}{r_{max}}$$

$$\int_0^\infty \frac{dk}{k} \to \int_{k_{min}}^{k_{max}} \frac{dk}{k} = \ln\left(\frac{k_{max}}{k_{min}}\right) = \ln\left(\frac{r_{max}}{r_{min}}\right) = L$$ ➔ **Coulomb logarithm**

Landau Formula, Coulomb logarithm

Plasma (gas) parameter:

$$\eta = \frac{U_{pot}}{E_k} \ll 1$$

$$\varkappa_{ij}{}^{\alpha\beta} = \frac{1}{8\pi}\left(\frac{e_\alpha e_\beta}{\varepsilon_0}\right)^2 \cdot \frac{u^2 - u_i u_j}{u^2} \cdot L$$

Coulomb logarithm $L = \ln\left(\frac{k_{max}}{k_{min}}\right) = \ln\left(\frac{r_{max}}{r_{min}}\right)$ – **Landau (1936)**

$$k_{min} = \frac{1}{r_D} \qquad k_{max} = \frac{1}{r_{min}} \cong \frac{\langle \mathcal{E} \rangle}{e^2}$$

$$L \gg 1 \Longleftarrow \frac{k_{min}}{k_{max}} \cong \frac{e^2 N^{\frac{1}{3}}}{\varepsilon_0 \langle \mathcal{E} \rangle} \ll 1$$

Gas-discharge, ionosphere	10…20
Semiconductor (nondegenerate)	10
Metals (degenerate) X	1

Landau equation

$\boldsymbol{u} = \boldsymbol{v}_{\alpha} - \boldsymbol{v}_{\beta}$

$$\frac{\partial f_\alpha}{\partial t} + \vec{v}\frac{\partial f_\alpha}{\partial \vec{r}} + e_\alpha\left(\vec{E} + [\vec{v} \times \vec{B}]\right)\frac{\partial f_\alpha}{\partial \vec{p}} =$$

$$= \sum_\beta \frac{\partial}{\partial p_{\alpha i}} \int d\boldsymbol{p}_{\boldsymbol{\beta}} \frac{1}{8\pi}\left(\frac{e_\alpha e_\beta}{\varepsilon_0}\right)^2 \cdot L \cdot \frac{u^2 - u_i u_j}{u^2} \cdot \left[\frac{\partial f_\alpha(\boldsymbol{p}_{\boldsymbol{\alpha}})}{\partial p_{\alpha j}} f_\beta(\boldsymbol{p}_{\boldsymbol{\beta}}) - f_\alpha(\boldsymbol{p}_{\boldsymbol{\alpha}}) \frac{\partial f_\beta(\boldsymbol{p}_\beta)}{\partial p_{\beta j}}\right]$$

Landau equation ➔ kinetic equation for a completely ionized plasma taking account of two-particle Coulomb collisions

Relaxation of the mean velocity

Problem 10.1: Calculate the relaxation of the mean velocity for electrons distributed according to a shifted Maxwellian in a completely ionized plasma.

$$f_e(\boldsymbol{v}) = \frac{N_e}{(2\pi mkT_e)^{3/2}} exp\left[-\frac{m(\boldsymbol{v}-\boldsymbol{u(t)})^2}{2\kappa T_e}\right]$$

Assumptions: $\boldsymbol{v_e} \gg \boldsymbol{v_i}$ and $\boldsymbol{v_{Te}} \gg \boldsymbol{v_{Ti}}$

$$\frac{\partial f_e}{\partial t} = \frac{\partial}{\partial v_s}\int d\boldsymbol{v^i}\frac{1}{8\pi}\left(\frac{ee_i}{m\varepsilon_0}\right)^2 \cdot L \cdot \frac{v^2 - v_s v_j}{v^2} \cdot \left[\frac{\partial f_e(\boldsymbol{v})}{\partial v_j} f_i(\boldsymbol{v^i}) - f_e(\boldsymbol{v})\frac{\partial f_i(\boldsymbol{v^i})}{\partial {v^i}_j}\right] \quad \times \boldsymbol{v} \int d\boldsymbol{p}$$

$$\frac{\partial \boldsymbol{u}}{\partial t} = -\frac{1}{8\pi}\left(\frac{ee_i}{m\varepsilon_0}\right)^2 \cdot \frac{L \cdot N_i}{N_e}\int d\boldsymbol{p} \cdot f_e \cdot \frac{2\boldsymbol{v}}{v^3}$$

Relaxation of the mean velocity

$$f_e(\boldsymbol{v}) = \frac{N_e}{(2\pi mkT_e)^{3/2}} exp\left[-\frac{m(\boldsymbol{v}-\boldsymbol{u}(\boldsymbol{t}))^2}{2\kappa T_e}\right]$$

$$\frac{\partial \boldsymbol{u}}{\partial t} = -\frac{1}{8\pi}\left(\frac{ee_i}{m\varepsilon_0}\right)^2 \cdot \frac{L \cdot N_i}{N_e}\int d\boldsymbol{p}\cdot f_e \cdot \frac{2\boldsymbol{v}}{v^3}$$

Assumptions: $\boldsymbol{u} \ll \boldsymbol{v_{Te}}$

$$\frac{\partial \boldsymbol{u}}{\partial t} = -\nu_{eff}\boldsymbol{u}$$

$$\nu_{eff} = \frac{1}{12\pi^2{\varepsilon_0}^2}\sqrt{\frac{2\pi}{m}}\cdot\frac{e^2{e_i}^2 L\cdot N_i}{(kT_e)^{3/2}}$$

⇨ Effective frequency of electron-ion collisions

$$T_{v-relax} \sim \frac{1}{\nu_{eff}}$$

➔ the relaxation time of the directed electrons

Particle collisions in the degenerate plasma

- Pauli's exclusion principle: at most two fermions (electrons or holes) can occupy the same state (→ $\boldsymbol{p}$)
- Each state in phase space has the volume $(2\pi\hbar)^3$

$$I_{\alpha\beta}(f_\alpha, f_\beta) = \frac{2\pi}{\hbar}\int d\boldsymbol{p}_{\boldsymbol{\beta}} \frac{d\boldsymbol{k}}{(2\pi)^3}|U(\boldsymbol{k})|^2 \cdot \delta\left(\frac{(\boldsymbol{p}_{\boldsymbol{\alpha}} - \hbar\boldsymbol{k})^2}{2m_{\boldsymbol{\alpha}}} + \frac{\left(\boldsymbol{p}_\beta + \hbar\boldsymbol{k}\right)^2}{2m_\beta} - \frac{\boldsymbol{p}_{\alpha}{}^2}{2m_{\boldsymbol{\alpha}}} - \frac{\boldsymbol{p}_{\beta}{}^2}{2m_\beta}\right) \times$$

$$\times\left[f_\alpha(\boldsymbol{p}_{\boldsymbol{\alpha}}) f_\beta(\boldsymbol{p}_{\boldsymbol{\beta}}) - f_\alpha(\boldsymbol{p}_{\boldsymbol{\alpha}} - \hbar\boldsymbol{k}) f_\beta(\boldsymbol{p}_\beta + \hbar\boldsymbol{k})\right]$$

$$\left[f_\alpha(\boldsymbol{p}_{\boldsymbol{\alpha}}) f_\beta(\boldsymbol{p}_{\boldsymbol{\beta}}) \cdot \left\{1 - \frac{(2\pi\hbar)^3}{2} f_\alpha(\boldsymbol{p}_{\boldsymbol{\alpha}} - \hbar\boldsymbol{k})\right\}\left\{1 - \frac{(2\pi\hbar)^3}{2} f_\beta(\boldsymbol{p}_\beta + \hbar\boldsymbol{k})\right\} - \right.$$

$$\left. - f_\alpha(\boldsymbol{p}_{\boldsymbol{\alpha}} - \hbar\boldsymbol{k}) f_\beta(\boldsymbol{p}_\beta + \hbar\boldsymbol{k}) \cdot \left\{1 - \frac{(2\pi\hbar)^3}{2} f_\alpha(\boldsymbol{p}_\alpha)\right\}\left\{1 - \frac{(2\pi\hbar)^3}{2} f_\beta(\boldsymbol{p}_\beta)\right\}\right]$$

Particle collisions in the degenerate plasma

NB: nondegenerate case: $I_{\alpha\beta}(f_\alpha, f_\beta) = \frac{\partial}{\partial p_{\alpha i}} \int d\boldsymbol{p_\beta}\ \varkappa_{ij}{}^{\alpha\beta}(\boldsymbol{p_\alpha}, \boldsymbol{p_\beta}) \left[\frac{\partial f_\alpha(\boldsymbol{p_\alpha})}{\partial p_{\alpha j}} f_\beta(\boldsymbol{p_\beta}) - f_\alpha(\boldsymbol{p_\alpha}) \frac{\partial f_\beta(\boldsymbol{p_\beta})}{\partial p_{\beta j}} \right]$

$\hbar \boldsymbol{k} \ll \boldsymbol{p_\alpha}, \boldsymbol{p_\beta}$

$$I_{\alpha\beta}(f_\alpha, f_\beta) = \\ = \int d\boldsymbol{p_\beta}\ \varkappa_{ij}{}^{\alpha\beta}(\boldsymbol{p_\alpha}, \boldsymbol{p_\beta}) \left[2 \frac{\partial f_\alpha}{\partial p_{\alpha i}} \frac{\partial f_\beta}{\partial p_{\beta j}} \left\{ 1 - \frac{(2\pi\hbar)^3}{2} (f_\alpha + f_\beta) \right\} - \right. \\ \left. - \frac{\partial^2 f_\alpha}{\partial p_{\alpha i} \partial p_{\alpha j}} f_\beta \left\{ 1 - \frac{(2\pi\hbar)^3}{2} f_\beta \right\} - \frac{\partial^2 f_\beta}{\partial p_{\beta i} \partial p_{\beta j}} f_\alpha \left\{ 1 - \frac{(2\pi\hbar)^3}{2} f_\alpha \right\} \right]$$

$$\varkappa_{ij}{}^{\alpha\beta}(\boldsymbol{p_\alpha}, \boldsymbol{p_\beta}) = \frac{\pi}{2} \cdot \frac{u^2 - u_i u_j}{u^2} \int (2\pi)^3 k^2\, d\boldsymbol{k} \cdot |U(\boldsymbol{k})|^2 \cdot \delta(\boldsymbol{ku})$$

$$kT_e \to \mathcal{E}_F$$

Summary

- Kinetic approach with particle collisions
  - Boltzmann collision integral – using QM approach and Born approximation
  - The Fokker-Planck equation – diffusion and friction in momentum space

$$\frac{\partial f_\alpha}{\partial t} + \vec{v}\frac{\partial f_\alpha}{\partial \vec{r}} + e_\alpha\left(\vec{E} + [\vec{v}\times\vec{B}]\right)\frac{\partial f_\alpha}{\partial \vec{p}} = \frac{\partial}{\partial p_{\alpha i}}\left(D_{ij}\frac{\partial f_\alpha}{\partial p_{\alpha j}} - A_i \cdot f_\alpha\right)$$

$$D_{ij} = \sum_\beta \int d\boldsymbol{p_\beta}\ \varkappa_{ij}{}^{\alpha\beta}(\boldsymbol{p_\alpha}, \boldsymbol{p_\beta}) f_\beta(\boldsymbol{p_\beta}); \qquad A_i = \sum_\beta \int d\boldsymbol{p_\beta}\ \varkappa_{ij}{}^{\alpha\beta}(\boldsymbol{p_\alpha}, \boldsymbol{p_\beta}) \frac{\partial f_\beta(\boldsymbol{p_\beta})}{\partial p_{\beta j}}$$

$$\varkappa_{ij}{}^{\alpha\beta}(\boldsymbol{p_\alpha}, \boldsymbol{p_\beta}) = \frac{\pi}{2}\cdot\frac{u^2 - u_i u_j}{u^2}\int (2\pi)^3 k^2\, d\boldsymbol{k}\cdot |U(\boldsymbol{k})|^2 \cdot \delta(\boldsymbol{ku})$$

  - Collision integral for charged particles
    - Lienard-Balescu integral → Dynamically screened kernel of the collision integral
    - $\varkappa_{ij}{}^{\alpha\beta} = \pi\left(\frac{e_\alpha e_\beta}{8\pi^3\varepsilon_0}\right)^2 \int (2\pi)^3\, d\boldsymbol{k}\cdot \frac{k_i k_j \delta(\boldsymbol{kv_\alpha} - \boldsymbol{kv_\beta})}{|k_i k_j \varepsilon_{ij}(\boldsymbol{kv_\alpha}, \boldsymbol{k})|^2}$
    - Landau Formula, Coulomb logarithm ➔ Landau equation
    - $\varkappa_{ij}{}^{\alpha\beta} = \frac{1}{8\pi}\left(\frac{e_\alpha e_\beta}{\varepsilon_0}\right)^2\cdot\frac{u^2 - u_i u_j}{u^2}\cdot\int_0^\infty \frac{dk}{k};$ $\qquad L = \ln\left(\frac{k_{max}}{k_{min}}\right) = \ln\left(\frac{r_{max}}{r_{min}}\right) = \ln\left(\frac{\varepsilon_0\langle\mathcal{E}\rangle}{e^2 N^{\frac{1}{3}}}\right)$

- Relaxation of the mean velocity:
  - effective frequency of electron-ion collisions in a completely ionized plasma: $\nu_{eff} = \frac{1}{12\pi^2\varepsilon_0{}^2}\sqrt{\frac{2\pi}{m}}\cdot\frac{e^2 e_i{}^2 L\cdot N_i}{(kT_e)^{3/2}}$
- Particle collisions in the degenerate plasma
  - Pauli's exclusion principle → $1 - \frac{(2\pi\hbar)^3}{2} f_\alpha$

- Next: Collision integrals for neutral particles:
  - → Model integral for elastic particle scattering → Bhatnagar-Gross-Krook =BGK

$$\left(\frac{\partial f_\alpha}{\partial t}\right)^{\alpha\beta} = -\nu_{\alpha\beta}\cdot\left(f_\alpha - N_\alpha \Phi_{\alpha\beta}\right)$$

$$\Phi_{\alpha\beta} = \frac{1}{(2\pi m_\alpha k T_{\alpha\beta})^{3/2}} exp\left[-\frac{m_\alpha(\boldsymbol{v} - \boldsymbol{V_\beta})^2}{2\kappa T_{\alpha\beta}}\right]; \qquad T_{\alpha\beta} = \frac{m_\alpha T_\beta + m_\beta T_\alpha}{m_\alpha + m_\beta}$$