Kinetic equation with particle collisions - 2

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L#11

Discussed last time…

- Kinetic approach with particle collisions
  - Boltzmann collision integral – using QM approach and Born approximation
  - The Fokker-Planck equation – diffusion and friction in momentum space

$$\frac{\partial f_\alpha}{\partial t} + \vec{v}\frac{\partial f_\alpha}{\partial \vec{r}} + e_\alpha\left(\vec{E} + [\vec{v}\times\vec{B}]\right)\frac{\partial f_\alpha}{\partial \vec{p}} = \frac{\partial}{\partial p_{\alpha i}}\left(D_{ij}\frac{\partial f_\alpha}{\partial p_{\alpha j}} - A_i \cdot f_\alpha\right)$$

$$D_{ij} = \sum_\beta \int d\boldsymbol{p_\beta}\ \varkappa_{ij}{}^{\alpha\beta}(\boldsymbol{p_\alpha}, \boldsymbol{p_\beta}) f_\beta(\boldsymbol{p_\beta}); \qquad A_i = \sum_\beta \int d\boldsymbol{p_\beta}\ \varkappa_{ij}{}^{\alpha\beta}(\boldsymbol{p_\alpha}, \boldsymbol{p_\beta}) \frac{\partial f_\beta(\boldsymbol{p_\beta})}{\partial p_{\beta j}}$$

$$\varkappa_{ij}{}^{\alpha\beta}(\boldsymbol{p_\alpha}, \boldsymbol{p_\beta}) = \frac{\pi}{2}\cdot\frac{u^2 - u_i u_j}{u^2}\int (2\pi)^3 k^2\, d\boldsymbol{k}\cdot|U(\boldsymbol{k})|^2\cdot\delta(\boldsymbol{ku})$$

  - Collision integral for charged particles
    - Lienard-Balescu integral → Dynamically screened kernel of the collision integral
    - $\varkappa_{ij}{}^{\alpha\beta} = \pi\left(\frac{e_\alpha e_\beta}{8\pi^3\varepsilon_0}\right)^2 \int (2\pi)^3\, d\boldsymbol{k}\cdot \frac{k_i k_j \delta(\boldsymbol{kv_\alpha} - \boldsymbol{kv_\beta})}{|k_i k_j \varepsilon_{ij}(\boldsymbol{kv_\alpha}, \boldsymbol{k})|^2}$
    - Landau Formula, Coulomb logarithm ➔ Landau equation
    - $\varkappa_{ij}{}^{\alpha\beta} = \frac{1}{8\pi}\left(\frac{e_\alpha e_\beta}{\varepsilon_0}\right)^2\cdot\frac{u^2 - u_i u_j}{u^3}\cdot\int_0^\infty \frac{dk}{k};$ $\qquad L = \ln\left(\frac{k_{max}}{k_{min}}\right) = \ln\left(\frac{r_{max}}{r_{min}}\right) = \ln\left(\frac{\varepsilon_0\langle\mathcal{E}\rangle}{e^2 N^{\frac{1}{3}}}\right)$
- Relaxation of the mean velocity:
  - effective frequency of electron-ion collisions in a completely ionized plasma: $\nu_{eff} = \frac{1}{12\pi^2\varepsilon_0{}^2}\sqrt{\frac{2\pi}{m}}\cdot\frac{e^2 e_i{}^2 L\cdot N_i}{(kT_e)^{3/2}}$
- Particle collisions in the degenerate plasma
  - Pauli's exclusion principle → $1 - \frac{(2\pi\hbar)^3}{2} f_\alpha$

- Next: Collision integrals for neutral particles:
  - → Model integral for elastic particle scattering → Bhatnagar-Gross-Krook =BGK

$$\left(\frac{\partial f_\alpha}{\partial t}\right)^{\alpha\beta} = -\nu_{\alpha\beta}\cdot\left(f_\alpha - N_\alpha \Phi_{\alpha\beta}\right)$$

$$\Phi_{\alpha\beta} = \frac{1}{(2\pi m_\alpha k T_{\alpha\beta})^{3/2}} exp\left[-\frac{m_\alpha(\boldsymbol{v} - \boldsymbol{V}_\beta)^2}{2\kappa T_{\alpha\beta}}\right]; \qquad T_{\alpha\beta} = \frac{m_\alpha T_\beta + m_\beta T_\alpha}{m_\alpha + m_\beta}$$

Collision frequency for charged particles

- effective frequency of electron-ion collisions in a completely ionized plasma:

$$\nu_{ei} \approx \nu_{eff} = \frac{1}{12\pi^2{\varepsilon_0}^2}\sqrt{\frac{2\pi}{m}} \cdot \frac{e^2{e_i}^2 L \cdot N_i}{(kT_e)^{3/2}}$$

- effective frequency of electron-electron collisions:

$$\nu_{ee} = \frac{1}{12\pi^2{\varepsilon_0}^2}\sqrt{\frac{\pi}{m}} \cdot \frac{e^4 L \cdot N_e}{(kT_e)^{3/2}}$$

- effective frequency of ion-ion collisions:

$$\nu_{ii} = \frac{1}{12\pi^2{\varepsilon_0}^2}\sqrt{\frac{\pi}{M}} \cdot \frac{e^4 L \cdot N_e}{(kT_i)^{3/2}}$$

$$\nu_{ie} \approx \nu_{ei} \cdot \frac{m}{M}\left|\frac{e_i}{e}\right|$$

L10: Relaxation of the mean velocity

Problem 10.1: Calculate the relaxation of the mean velocity for electrons distributed according to a shifted Maxwellian in a completely ionized plasma.

$$f_e(\boldsymbol{v}) = \frac{N_e}{(2\pi m k T_e)^{3/2}} exp\left[-\frac{m(\boldsymbol{v}-\boldsymbol{u}(\boldsymbol{t}))^2}{2\kappa T_e}\right]$$

$$\frac{\partial f_e}{\partial t} = \frac{\partial}{\partial v_s}\int d\boldsymbol{v^i}\,\frac{1}{8\pi}\left(\frac{ee_i}{m\varepsilon_0}\right)^2 \cdot L \cdot \frac{v^2 - v_s v_j}{v^2} \cdot \left[\frac{\partial f_e(\boldsymbol{v})}{\partial v_j} f_i(\boldsymbol{v^i}) - f_e(\boldsymbol{v})\frac{\partial f_i(\boldsymbol{v^i})}{\partial v^i{}_j}\right] \qquad \times\, \boldsymbol{v} \int d\boldsymbol{p}$$

$$\frac{\partial \boldsymbol{u}}{\partial t} = -\frac{1}{8\pi}\left(\frac{ee_i}{m\varepsilon_0}\right)^2 \cdot \frac{L \cdot N_i}{N_e}\int d\boldsymbol{p} \cdot f_e \cdot \frac{2\boldsymbol{v}}{v^3}$$

Assumptions: $\boldsymbol{u} \ll \boldsymbol{v_{Te}}$

$$\frac{\partial \boldsymbol{u}}{\partial t} = -\nu_{eff}\boldsymbol{u}$$

$$\nu_{eff} = \frac{1}{12\pi^2\varepsilon_0{}^2}\sqrt{\frac{2\pi}{m}} \cdot \frac{e^2 e_i{}^2 L \cdot N_i}{(kT_e)^{3/2}}$$

⇨ Effective frequency of electron-ion collisions

$$T_{v-relax} \sim \frac{1}{\nu_{eff}}$$ ➔ the relaxation time of the directed electrons

Completely ionized plasma in electric field

Problem 11.1: Calculate the electron distribution for a completely ionized plasma in a constant external electric field

Assumptions: Lorentz gas → electrons interact with ions only $Z = \left|\frac{e_i}{e}\right| \gg 1$

$$e\boldsymbol{E}\frac{\partial f_e}{\partial \boldsymbol{p}} = \frac{\partial}{\partial p_s}\frac{1}{8\pi}\left(\frac{ee_i}{\varepsilon_0}\right)^2 N_i L \frac{v^2\delta_{sj} - v_s v_j}{v^3}\frac{\partial f_e}{\partial p_j}$$

The solution → expansion: $f_e = f_M + \frac{\boldsymbol{v}\boldsymbol{f_1}}{v}$ $\quad |\boldsymbol{f_1}| \ll f_M$

$$e\boldsymbol{E}\frac{\partial f_M}{\partial \boldsymbol{p}} = \frac{1}{4\pi}\left(\frac{ee_i}{\varepsilon_0}\right)^2 \frac{N_i L}{m^2}\frac{\boldsymbol{v}\boldsymbol{f_1}}{m^4}$$

$$f_e = f_M + \frac{e\boldsymbol{E}}{m\nu(v)}\frac{\partial f_M}{\partial \boldsymbol{v}}$$

$$\nu(v) = \frac{1}{4\pi}\left(\frac{ee_i}{\varepsilon_0}\right)^2 \frac{N_i L}{m^2 v^3}$$

Completely ionized plasma in electric field

$$f_e = f_M + \frac{e\boldsymbol{E}}{m\nu(v)}\frac{\partial f_M}{\partial \boldsymbol{v}}$$

$$\nu(v) = \frac{1}{4\pi}\left(\frac{ee_i}{\varepsilon_0}\right)^2 \frac{N_i L}{m^2 v^3}$$

$$\boldsymbol{j} = e\int \boldsymbol{v} f_e \, d\boldsymbol{p} = \frac{32}{3\pi}\frac{e^2 N_e}{m\nu_{eff}}\boldsymbol{E}$$

$$\nu_{eff} = \frac{1}{12\pi^2\varepsilon_0{}^2}\sqrt{\frac{2\pi}{m}} \cdot \frac{e^2 e_i{}^2 L \cdot N_i}{(kT_e)^{3/2}}$$

$$\sigma = \frac{32}{3\pi}\frac{e^2 N_e}{m\nu_{eff}}$$

Plasma conductivity in the Lorentz gas model

Valid if $u = \frac{eE}{m\nu_{eff}} < v_{Te}$ → stationary solution

- if $u > v_{Te}$ → runaway electrons
- Dreicer field $E_{cr} = \frac{m\nu_{eff} \cdot v_{Te}}{e}$

Completely ionized plasma in electric field

Problem 11.1a: Calculate the electron distribution for a completely ionized plasma in a constant external electric field including **electron-electron** collisions

- Electrons → main contribution to the induced current
- Ions → unperturbed

$$e\boldsymbol{E}\frac{\partial f_e}{\partial \boldsymbol{p}} = \frac{\partial}{\partial p_s}\sum_{\alpha=e,i}\int d\boldsymbol{p}^{\boldsymbol{\alpha}}\frac{1}{8\pi}\left(\frac{ee_{\boldsymbol{\alpha}}}{\varepsilon_0}\right)^2\cdot L\cdot\frac{u^2-u_s u_j}{u^3}\cdot\left[f_\alpha\cdot\frac{\partial f_e}{\partial p_j}-f_e\cdot\frac{\partial f_\alpha}{\partial p^\alpha{}_j}\right]$$

$$\boldsymbol{u} = \boldsymbol{v} - \boldsymbol{v}^{\boldsymbol{\alpha}}$$

- Linearization $f_e = f_{e0} + \delta f_e$

$$e\boldsymbol{E}\frac{\partial f_{0e}}{\partial \boldsymbol{p}} = N_i\frac{\partial}{\partial p_s}\varkappa_{sj}{}^{ei}(\boldsymbol{p})\frac{\partial \delta f_e}{\partial p_j} +$$

$$+\frac{\partial}{\partial p_s}\int d\boldsymbol{p}'\varkappa_{sj}{}^{ee}(\boldsymbol{p},\boldsymbol{p}')\cdot\left[\frac{\partial f_{0e}}{\partial p_j}\delta f_e(\boldsymbol{p}')+\frac{\partial \delta f_e}{\partial p_j}f_{0e}(\boldsymbol{p}')\cdot\frac{\partial f_e}{\partial p_j}-f_{0e}(\boldsymbol{p})\frac{\partial \delta f_e}{\partial p'_j}-\delta f_e(\boldsymbol{p})\cdot\frac{\partial f_{0e}}{\partial p'_j}\right]$$

$$\varkappa_{ij}{}^{e\alpha}(\boldsymbol{p}) = \frac{1}{8\pi}\left(\frac{e\,e_\alpha}{\varepsilon_0}\right)^2\cdot\frac{u^2-u_i u_j}{u^3}\cdot L$$

Completely ionized plasma in electric field

Problem 11.1a: Calculate the electron distribution for a completely ionized plasma in a constant external electric field including **electron-electron** collisions

$$e\boldsymbol{E}\frac{\partial f_{0e}}{\partial \boldsymbol{p}} = N_i \frac{\partial}{\partial p_s}\varkappa_{sj}{}^{ei}(\boldsymbol{p})\frac{\partial \delta f_e}{\partial p_j} + \frac{\partial}{\partial p_s}\int d\boldsymbol{p}'\varkappa_{sj}{}^{ee}(\boldsymbol{p},\boldsymbol{p}')\cdot\left[\frac{\partial f_{0e}}{\partial p_j}\delta f_e(\boldsymbol{p}') + \frac{\partial \delta f_e}{\partial p_j} f_{0e}(\boldsymbol{p}')\cdot\frac{\partial f_e}{\partial p_j} - f_{0e}(\boldsymbol{p})\frac{\partial \delta f_e}{\partial p'_j} - \delta f_e(\boldsymbol{p})\cdot\frac{\partial f_{0e}}{\partial p'_j}\right]$$

$$\varkappa_{ij}{}^{e\alpha}(\boldsymbol{p}) = \frac{1}{8\pi}\left(\frac{e\,e_\alpha}{\varepsilon_0}\right)^2\cdot\frac{u^2 - u_i u_j}{u^3}\cdot L$$

To solve ➔ the Chapman–Enskog method = expanding δf_e in Sonin polynomials

$$\delta f_e(\boldsymbol{p}) = \frac{\boldsymbol{v}\cdot\boldsymbol{E}}{v}\left[a_0 + a_1\cdot\left(\frac{5}{2} - \frac{1}{2}\left(\frac{v}{v_{Te}}\right)^2\right) + \cdots\right] f_{0e}$$

$\times\,\mathbf{1}\ \int d\boldsymbol{p}$ → $\frac{eE}{2\kappa T_e} = -\nu_{eff}\cdot\left(a_0 + \frac{3}{2}a_1\right)$

$\times\left(\frac{5}{2} - \frac{1}{2}\left(\frac{v}{v_{Te}}\right)^2\right)\int d\boldsymbol{p}$ → $\frac{3}{2}a_0 + \frac{13+4\sqrt{2}}{4}a_1 = 0$

⇒

$$a_0 = -\frac{13+4\sqrt{2}}{6}a_1$$

$$a_1 = -\frac{3}{2+2\sqrt{2}}\cdot\frac{eE}{\kappa T_e \nu_{eff}}$$

Completely ionized plasma in electric field

Problem 11.1a: Calculate the electron distribution for a completely ionized plasma in a constant external electric field including **electron-electron** collisions

$$\delta f_e(\boldsymbol{p}) = \frac{\boldsymbol{v}\cdot\boldsymbol{E}}{v}\left[a_0 + a_1\cdot\left(\frac{5}{2}-\frac{1}{2}\left(\frac{v}{v_{Te}}\right)^2\right)+\cdots\right]f_{0e}$$

$$a_0 = -\frac{13+4\sqrt{2}}{6}a_1$$

$$a_1 = -\frac{3}{2+2\sqrt{2}}\cdot\frac{eE}{\kappa T_e \nu_{eff}}$$

$$\boldsymbol{j} = e\int \boldsymbol{v}\,\delta f_e\, d\boldsymbol{p} \approx 1.96\frac{e^2 N_e}{m\nu_{eff}}\boldsymbol{E}$$

Spitzer formula for the conductivity of a completely ionized plasma

$$\boldsymbol{\varepsilon}^{l} = \boldsymbol{\varepsilon}^{tr} = 1 + i\cdot 1.96\frac{{\omega_{pe}}^2}{\omega\nu_{eff}}$$

Temperature (energy) relaxation in a completely ionized plasma

Problem 11.2: Assuming Maxwell distributions of electrons (T_e) and ions (T_i) estimate the time of the temperature relaxation in a completely ionized plasma

$$f_e = \frac{N_e}{(2\pi m \kappa T_e)^{3/2}} e^{-\frac{mv^2}{2\kappa T_e}}$$

$$f_i = \frac{N_i}{(2\pi M \kappa T_i)^{3/2}} e^{-\frac{Mv^2}{2\kappa T_i}}$$

$$\left(\frac{\partial f_e}{\partial t}\right)^{ee} = \left(\frac{\partial f_i}{\partial t}\right)^{ii} = 0$$

$$\frac{\partial f_e}{\partial t} = \left(\frac{\partial f_e}{\partial t}\right)^{ei}$$

$$\frac{\partial f_e}{\partial t} = \frac{\partial}{\partial v_s} \int d\boldsymbol{v^i} \frac{1}{8\pi}\left(\frac{ee_i}{m\varepsilon_0}\right)^2 \cdot L \cdot \frac{v^2 - v_s v_j}{v^2} \cdot \left[\frac{\partial f_e(\boldsymbol{v})}{\partial v_j} f_i(\boldsymbol{v^i}) - f_e(\boldsymbol{v}) \frac{\partial f_i(\boldsymbol{v^i})}{\partial {v^i}_j}\right]$$

$$\times \frac{mv^2}{2} \int d\boldsymbol{p}^e$$

$$\frac{\partial T_e}{\partial t} = -\nu_T (T_e - T_i)$$

$$\nu_T = \frac{1}{12\pi}\left(\frac{ee_i}{\varepsilon_0}\right)^2 \frac{L}{\kappa T_i \kappa T_e} \frac{N_i}{(2\pi)^3 (m\kappa T_e M \kappa T_i)^{3/2}} \int d\boldsymbol{p}^e d\boldsymbol{p}^i\, e^{-\frac{m(v^e)^2}{2\kappa T_e} - \frac{m(v^i)^2}{2\kappa T_i}} \cdot {v^e}_s\, {v^e}_j \frac{u^2 \delta_{sj} - u_s u_j}{u^3}$$

$$\boldsymbol{u} = \boldsymbol{v^e} - \boldsymbol{v^i}$$

Temperature (energy) relaxation in a completely ionized plasma

Problem 11.2: Assuming Maxwell distributions of electrons (T_e) and ions (T_i) estimate the time of the temperature relaxation in a completely ionized plasma

$$\frac{\partial T_e}{\partial t} = -\nu_T (T_e - T_i)$$

$$\nu_T = \frac{1}{12\pi}\left(\frac{e e_i}{\varepsilon_0}\right)^2 \frac{L}{\kappa T_i \kappa T_e} \frac{N_i}{(2\pi)^3 (m\kappa T_e M\kappa T_i)^{3/2}} \int d\mathbf{p}^e d\mathbf{p}^i\, e^{-\frac{m(v^e)^2}{2\kappa T_e} - \frac{m(v^i)^2}{2\kappa T_i}} \cdot v^e{}_s\ v^e{}_j \frac{u^2 \delta_{sj} - u_s u_j}{u^3}$$

$$\mu = \frac{v_{Ti}}{v_{Te}} \ll 1 \Rightarrow \nu_T = 2\frac{m}{M}\nu_{eff}$$

$$\nu_{eff} = \frac{1}{12\pi^2 \varepsilon_0{}^2} \sqrt{\frac{2\pi}{m}} \cdot \frac{e^2 e_i{}^2 L \cdot N_i}{(kT_e)^{3/2}}$$

Total energy conservation: $N_e T_e + N_i T_i = const \rightarrow N_e \frac{dT_e}{dt} + N_i \frac{dT_i}{dt} = 0$

$$\frac{\partial (T_e - T_i)}{\partial t} = -\nu_T \left(1 + \left|\frac{e_i}{e}\right|\right)(T_e - T_i)$$

$t_{T-relax} \sim \frac{M}{m} \cdot \frac{1}{\nu_{eff}} \rightarrow$ the relaxation time of the temperature

Model Integral for elastic particle collisions

$$\frac{\partial f_\alpha}{\partial t} + \vec{v}\frac{\partial f_\alpha}{\partial \vec{r}} + e_\alpha\left(\vec{E} + [\vec{v} \times \vec{B}]\right)\frac{\partial f_\alpha}{\partial \vec{p}} = \sum_\beta \left(\frac{\partial f_\alpha}{\partial t}\right)^{\alpha\beta}$$

Any collision integral of elastic particle scattering → conservation:

- Number of particles: $\int d\boldsymbol{p} \left(\frac{\partial f_\alpha}{\partial t}\right)^{\alpha\beta} = 0$
- Momentum: $m_\alpha \int d\boldsymbol{p}\ \boldsymbol{v} \left(\frac{\partial f_\alpha}{\partial t}\right)^{\alpha\beta} + m_\beta \int d\boldsymbol{p}\ \boldsymbol{v} \left(\frac{\partial f_\beta}{\partial t}\right)^{\beta\alpha} = 0$
- Energy: $m_\alpha \int d\boldsymbol{p}\ \boldsymbol{v}^2 \left(\frac{\partial f_\alpha}{\partial t}\right)^{\alpha\beta} + m_\beta \int d\boldsymbol{p}\ \boldsymbol{v}^2 \left(\frac{\partial f_\beta}{\partial t}\right)^{\beta\alpha} = 0$

Boltzmann H-Theorem: the increase in the entropy of an ideal gas in an irreversible process. Collisions destroy any perturbation of the equilibrium distribution the Maxwell or Fermi-Dirac)

Maxwell, Fermi-Dirac → $\left(\frac{\partial f_\alpha}{\partial t}\right)^{\alpha\beta} \to 0$

Model Integral for elastic particle collisions

for elastic particle scattering in a nondegenerate weakly ionized plasma
1954 Bhatnagar-Gross-Krook = BGK integral:

$$\left(\frac{\partial f_\alpha}{\partial t}\right)^{\alpha\beta} = -\nu_{\alpha\beta} \cdot \left(f_\alpha - N_\alpha \Phi_{\alpha\beta}\right)$$

$$\Phi_{\alpha\beta} = \frac{1}{\left(2\pi m_\alpha k T_{\alpha\beta}\right)^{3/2}} exp\left[-\frac{m_\alpha\left(\boldsymbol{v}-\boldsymbol{V_\beta}\right)^2}{2\kappa T_{\alpha\beta}}\right]$$ → local Maxwell distribution

$$N_\alpha = \int f_\alpha \, d\boldsymbol{p} \; ; \qquad \boldsymbol{V}_\alpha = \frac{1}{N_\alpha}\int \boldsymbol{v} f_\alpha \, d\boldsymbol{p}$$

$$T_{\alpha\beta} = \frac{m_\alpha T_\beta + m_\beta T_\alpha}{m_\alpha + m_\beta}$$

$$T_\alpha = \frac{m_\alpha}{2N_\alpha}\int (\boldsymbol{v} - \boldsymbol{V}_\alpha)^2 f_\alpha \, d\boldsymbol{p}$$

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A Model for Collision Processes in Gases. I. Small Amplitude Processes in Charged and Neutral One-Component Systems*

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(Received November 27, 1953)

BGK Integral for elastic particle collisions

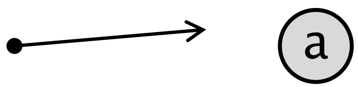
- Momentum: $m_\alpha \int d\boldsymbol{p}\ \boldsymbol{v} \left(\frac{\partial f_\alpha}{\partial t}\right)^{\alpha\beta} + m_\beta \int d\boldsymbol{p}\ \boldsymbol{v} \left(\frac{\partial f_\beta}{\partial t}\right)^{\beta\alpha} = 0$
- Energy: $m_\alpha \int d\boldsymbol{p}\ \boldsymbol{v}^2 \left(\frac{\partial f_\alpha}{\partial t}\right)^{\alpha\beta} + m_\beta \int d\boldsymbol{p}\ \boldsymbol{v}^2 \left(\frac{\partial f_\beta}{\partial t}\right)^{\beta\alpha} = 0$

$$m_\alpha N_\alpha \nu_{\alpha\beta} = m_\beta N_\beta \nu_{\beta\alpha}$$

$$\left(\frac{\partial f_\alpha}{\partial t}\right)^{\alpha\beta} = -\nu_{\alpha\beta} \cdot \left(f_\alpha - N_\alpha \Phi_{\alpha\beta}\right)$$

$\nu_{\alpha n} - ?$

From simple molecular-kinetic considerations:



$$\nu_{\alpha n} = \pi a^2 v_{T\alpha} N_n$$

$$a \sim 10^{-10} m$$

BGK Integral for degenerate plasma

$$\left(\frac{\partial f_\alpha}{\partial t}\right)^{\alpha\beta} = -\nu_{\alpha\beta}\cdot(f_\alpha - f_{0\alpha})$$

Fermi distribution function $f_{0\alpha}(\boldsymbol{p}_\alpha) = f_{F\alpha} = \dfrac{\frac{2}{(2\pi\hbar)^3}}{exp\left(\frac{\frac{p_\alpha{}^2}{2m_\alpha} - \varepsilon_{F\alpha}}{\kappa T_\alpha}\right)+1}$

$$\varepsilon_{F\alpha} = \frac{p_{F\alpha}{}^2}{2m_\alpha} = \frac{(3\pi^2)^{2/3}\hbar^2 N_\alpha{}^{2/3}}{2m_\alpha}$$

$$\nu_{\alpha n} \sim \frac{1}{\tau_{\alpha n}}$$

particle life time (from experiment)

Relaxation of the mean velocity

Problem 11.2: Calculate the relaxation of the mean velocity for electrons distributed according to a shifted Maxwell distribution in a weakly ionized plasma.

$$f_e(\boldsymbol{v}) = \frac{N_e}{(2\pi mkT_e)^{3/2}} exp\left[-\frac{m(\boldsymbol{v}-\boldsymbol{u}(\boldsymbol{t}))^2}{2\kappa T_e}\right]$$

Assumptions: $m \ll M$

$$T_{en} = \frac{m_e T_n + M T_e}{m_e + M} \approx T_e$$

$$\frac{\partial f_e}{\partial t} = \left(\frac{\partial f_\alpha}{\partial t}\right)^{en} = -\nu_{en} \cdot (f_e - N_e \Phi_{en})$$

$\times \boldsymbol{v} \int d\boldsymbol{p}$

$$\Phi_{en} = \frac{1}{(2\pi m_e k T_{en})^{3/2}} exp\left[-\frac{m_\alpha(\boldsymbol{v}-\boldsymbol{V_n})^2}{2\kappa T_{\alpha\beta}}\right] \xrightarrow{m \ll M} \frac{1}{(2\pi m_e k T_e)^{3/2}} exp\left[-\frac{m_\alpha \boldsymbol{v}^2}{2\kappa T_e}\right]$$

$$\frac{\partial \boldsymbol{u}}{\partial t} = -\nu_{en}\boldsymbol{u}$$

$T_{v-relax} \sim \frac{1}{\nu_{en}}$ ➔ the relaxation time of the directed electrons

Weakly ionized plasma in electric field

Problem 11.3: Calculate the electron distribution for a weakly ionized plasma in a constant external electric field

$$e\boldsymbol{E}\frac{\partial f_e}{\partial \boldsymbol{p}} = -\nu_{en} \cdot (f_e - N_e \Phi_{en})$$

$$\Phi_{en} = \frac{1}{(2\pi mkT_{en})^{3/2}} exp\left[-\frac{m\boldsymbol{v}^2}{2\kappa T_{en}}\right]$$

$$T_{en} = \frac{m_e T_n + MT_e}{m_e + M}$$

The solution → expansion: $f_e = f_0 + \frac{\boldsymbol{v}\boldsymbol{f_1}}{v}$ $\quad |\boldsymbol{f_1}| \ll f_0$

$$\frac{e}{3m\,v^2}\frac{\partial}{\partial v}(v^2 \boldsymbol{E}\cdot\boldsymbol{f_1}) = -\nu_{en} \cdot (f_0 - N_e\Phi_{en})$$

$$\frac{e\boldsymbol{E}}{m}\frac{\partial f_0}{\partial v} = -\nu_{en} \cdot \boldsymbol{f_1}$$

Weakly ionized plasma in electric field

Problem 11.3: Calculate the electron distribution for a weakly ionized plasma in a constant external electric field

$$\frac{e}{3m\,v^2}\frac{\partial}{\partial v}\left(v^2 \boldsymbol{E}\cdot\boldsymbol{f_1}\right) = -\nu_{en}\cdot\left(f_0 - N_e\Phi_{en}\right)$$

$$\frac{e\boldsymbol{E}}{m}\frac{\partial f_0}{\partial v} = -\nu_{en}\cdot\boldsymbol{f_1}$$

$$\frac{e^2E^2}{3m\nu_{en}}\frac{1}{v^2}\frac{\partial}{\partial v}\left(v^2\frac{\partial f_0}{\partial v}\right) + \nu_{en}\cdot\left(f_0 - N_e\Phi_{en}\right) = 0$$

$$f_0 = \frac{1}{(2\pi mkT_e)^{3/2}}\,exp\left[-\frac{mv^2}{2\kappa T_e}\right]$$

Weakly ionized plasma in electric field

Problem 11.3: Calculate the electron distribution for a weakly ionized plasma in a constant external electric field

$$\frac{e^2E^2}{3m\nu_{en}}\frac{1}{v^2}\frac{\partial}{\partial v}\left(v^2\frac{\partial f_0}{\partial v}\right)+\nu_{en}\cdot\left(f_0-N_e\Phi_{en}\right)=0$$

$$f_0=\frac{1}{(2\pi mkT_e)^{3/2}}exp\left[-\frac{m\boldsymbol{v}^2}{2\kappa T_e}\right]$$

$$\frac{2e^2E^2}{m^2\nu_{en}}-\nu_{en}\cdot\frac{3}{m+M}\left(T_e-T_n\right)=0$$

Stationary solution: balance between the ohmic electron heating and the energy transfer e→n

$$T_e\approx T_n+\frac{2}{3}\frac{M}{m}\frac{e^2E^2}{m\,{\nu_{en}}^2}$$

Weakly ionized plasma in electric field

Problem 11.3: Calculate the electron distribution for a weakly ionized plasma in a constant external electric field

$$f_e = f_0 + \frac{\boldsymbol{v} \boldsymbol{f_1}}{v}$$

$$\boldsymbol{f_1} = -\frac{e\boldsymbol{E}}{m\nu_{en}} \frac{\partial f_0}{\partial v}$$

$$f_0 = \frac{1}{(2\pi mkT_e)^{3/2}} exp\left[-\frac{m\boldsymbol{v}^2}{2\kappa T_e}\right]$$

$$\boldsymbol{j} = e \int \boldsymbol{v}\, f_e\, d\boldsymbol{p} = e \int \boldsymbol{v} \frac{\boldsymbol{v} \boldsymbol{f_1}}{v} d\boldsymbol{p}$$

$$\boldsymbol{j} = \frac{e^2 N_e}{m\nu_{en}} \boldsymbol{E}$$

$$\boldsymbol{\varepsilon}^{\boldsymbol{l}} = \boldsymbol{\varepsilon}^{\boldsymbol{tr}} = 1 + i \cdot \frac{{\omega_{pe}}^2}{\omega \nu_{en}}$$

Electron collision frequency ν_e

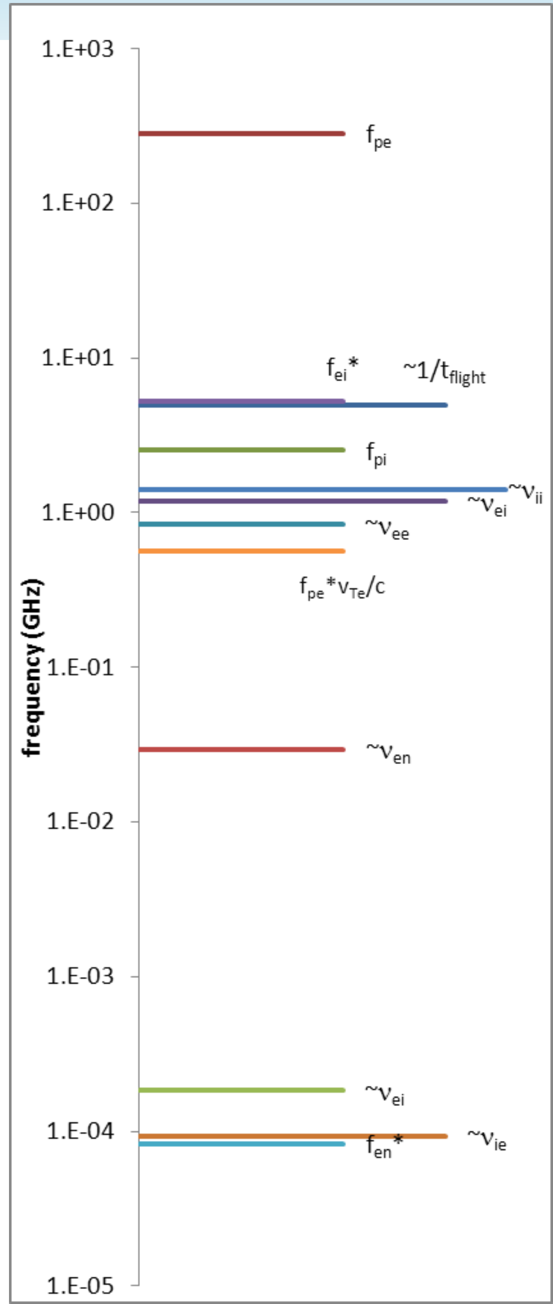
Plasma → (↓)	Nondegenerate	Degenerate
Completely ionized	$\nu_{eff} = \frac{1}{12\pi^2{\varepsilon_0}^2}\sqrt{\frac{2\pi}{m}} \cdot \frac{e^2{e_i}^2L \cdot N_i}{(kT_e)^{3/2}}$	$\nu_{Fe} = \frac{1}{4\pi{\varepsilon_0}^2}\sqrt{\frac{2\pi}{m}} \cdot \frac{e^2{e_i}^2L \cdot N_i}{{\mathcal{E}_F}^{3/2}}$
Weakly ionized	$\nu_{\alpha n} = \pi a^2 v_{T\alpha} N_n$ $a \sim 10^{-10} m$	

PWA at PITZ: frequencies

			electrons		ions		Li atoms	
mass	m,M	kg	9.1E-31		1.2E-26		1.2E-26	
density / distance	N	cm^{-3}	1.0E+15		1.0E+15		1.0E+16	
		m^{-3}	1.0E+21		1E+21		1E+22	
	r_L	m	1.0E-07		1.0E-07		4.6E-08	
		µm	0.1		0.1		0.05	
frequency / wavelength	ω_p	s^{-1}	1.8E+12		1.6E+10			
	f_p	Hz	2.8E+11		2.5E+09			
		GHz	284		3	->TDS?		
	$\lambda_{p=}c/fp$	m	1.1E-03		1.2E-01			
		mm	1.1		119			
thermal	T	eV	2.0E+00		0.06		0.06	
		K	2.3E+04		700		700	
	kT	J	3.2E-19		9.7E-21		9.7E-21	
	v_T	m/s	5.9E+05		916		9[illegible]	
Debye	r_D	m	3.3E-07		5.8E-08			
		µm	0.33		0.06			
Fermi	E_F	J	5.8E-24	-> classical	4.6E-28	-> classical		
		eV	3.6E-05		2.9E-09			
gas par.	η	-	7.2E-03	->gas	0.24	~>gas		
collisions	ν_{ei}	s^{-1}	7.4E+09					
	ν_{ee}	s^{-1}	5.3E+09					
	ν_{ie}	s^{-1}			5.9E+05			
	ν_{ii}	s^{-1}			8.9E+09			
	ν_{en}	s^{-1}	1.9E+08					
	ν_{in}	s^{-1}			1.2E+06			

$$t_{flight} = \frac{6cm}{v_{beam}}$$

freq	GHz
$\sim 1/t_{flight}$	5.0
f_{pe}	284
f_{pi}	2.5
$\nu_{ei}/(2\pi)$	1.2
$\nu_{ee}/(2\pi)$	0.84
$\nu_{ie}/(2\pi)$	0.0001
$\nu_{ii}/(2\pi)$	1.42
$\nu_{en}/(2\pi)$	0.030
$\nu_{ei}/(2\pi)$	0.0002
f_{ei}*	5.2
f_{en}*	0.0001
$f_{pe}*v_{Te}/c$	0.56



PWA at PITZ: times

			electrons		ions		Li atoms	
mass	m,M	kg	9.1E-31		1.2E-26		1.2E-26	
density / distance	N	cm^{-3}	1.0E+15		1.0E+15		1.0E+16	
		m^{-3}	1.0E+21		1E+21		1E+22	
	r_L	m	1.0E-07		1.0E-07		4.6E-08	
		µm	0.1		0.1		0.05	
frequency / wavelength	ω_p	s^{-1}	1.8E+12		1.6E+10			
	f_p	Hz	2.8E+11		2.5E+09			
		GHz	284		3	->TDS?		
	$\lambda_{p=}c/fp$	m	1.1E-03		1.2E-01			
		mm	1.1		119			
thermal	T	eV	2.0E+00		0.06		0.06	
		K	2.3E+04		700		700	
	kT	J	3.2E-19		9.7E-21		9.7E-21	
	v_T	m/s	5.9E+05		916		916	
Debye	r_D	m	3.3E-07		5.8E-08			
		µm	0.33		0.06			
Fermi	E_F	J	5.8E-24	-> classical	4.6E-28	-> classical		
		eV	3.6E-05		2.9E-09			
gas par.	η	-	7.2E-03	->gas	0.24	~>gas		
collisions	ν_{ei}	s^{-1}	7.4E+09					
	ν_{ee}	s^{-1}	5.3E+09					
	ν_{ie}	s^{-1}			5.9E+05			
	ν_{ii}	s^{-1}			8.9E+09			
	ν_{en}	s^{-1}	1.9E+08					
	ν_{in}	s^{-1}			1.2E+06			

time	(ps)	in t_{flight}
t_{flight}	**200**	**1**
t_{pe}	**4**	**0.018**
t_{pi}	**396**	**2.0**
τ_{ei}	**846**	**4.2**
τ_{ee}	**1196**	**6.0**
τ_{ie}	1E+07	5E+04
τ_{ii}	**706**	**3.5**
τ_{en}	3E+04	2E+02
τ_{ei}	5E+06	3E+04
t_{ei}*	**191**	**1.0**
t_{en}*	1E+07	6E+04
$t_{ei(T\text{-}relax)}$	1E+07	5E+04
$t_{en(T\text{-}relax)}$	2E+07	8E+04

$$t_{flight} = \frac{6cm}{v_{beam}}$$

Summary

- Kinetic approach with particle collisions
  - Effective frequency of electron-ion collisions in a completely ionized plasma: $\nu_{eff} = \frac{1}{12\pi^2\varepsilon_0{}^2}\sqrt{\frac{2\pi}{m}} \cdot \frac{e^2 e_i{}^2 L \cdot N_i}{(kT_e)^{\frac{3}{2}}}$ (v-relax)
  - Completely ionized plasma in electric field:
    - Lorentz gas model → plasma conductivity: $\sigma = \frac{32}{3\pi}\frac{e^2 N_e}{m\nu_{eff}}$
    - Including electron-electron collisions → Spitzer formula $\sigma = 1.96\frac{e^2 N_e}{m\nu_{eff}}$ ➔ $\boldsymbol{\varepsilon}^{\boldsymbol{l}} = \boldsymbol{\varepsilon}^{\boldsymbol{tr}} = 1 + i \cdot 1.96\frac{\omega_{pe}{}^2}{\omega\nu_{eff}}$
    - Temperature (energy) relaxation in a completely ionized plasma: $t_{T-relax} \sim \frac{M}{m} \cdot \frac{1}{\nu_{eff}}$
  - Integrals for collisions of charged and neutral particles:
    - Model integral for elastic particle scattering → Bhatnagar-Gross-Krook =BGK (e-n; i-n)

$$\left(\frac{\partial f_\alpha}{\partial t}\right)^{\alpha\beta} = -\nu_{\alpha\beta} \cdot (f_\alpha - N_\alpha \Phi_{\alpha\beta}); \quad \Phi_{\alpha\beta} = \frac{1}{(2\pi m_\alpha k T_{\alpha\beta})^{3/2}} exp\left[-\frac{m_\alpha(\boldsymbol{v}-\boldsymbol{V}_\beta)^2}{2\kappa T_{\alpha\beta}}\right]; \quad T_{\alpha\beta} = \frac{m_\alpha T_\beta + m_\beta T_\alpha}{m_\alpha + m_\beta}$$

Frequency of collisions: $\nu_{\alpha n} = \pi a^2 v_{T\alpha} N_n \quad a \sim 10^{-10} m$

- BGK integrals for degenerate plasma: $\Phi_{\alpha\beta} \to f_{F\alpha} = \frac{\frac{2}{(2\pi\hbar)^3}}{exp\left(\frac{\frac{p_\alpha{}^2}{2m_\alpha} - \varepsilon_{F\alpha}}{\kappa T_\alpha}\right) + 1}$
- Relaxation of the mean velocity: $T_{v-relax} \sim \frac{1}{\nu_{en}}$ (v-relax)
- Weakly ionized plasma in electric field:

$$T_e \approx T_n + \frac{2}{3}\frac{M}{m}\frac{e^2 E^2}{m\,\nu_{en}{}^2}$$ ➔ stationary solution

plasma conductivity: $\sigma = \frac{e^2 N_e}{m\nu_{en}}$ ➔ $\boldsymbol{\varepsilon}^{\boldsymbol{l}} = \boldsymbol{\varepsilon}^{\boldsymbol{tr}} = 1 + i \cdot \frac{\omega_{pe}{}^2}{\omega\nu_{en}}$

- PWA at PITZ → plasma: strongly ionized, almost collisionless, nonisothermal

- Next → plasma skin-depth vs. frequency