6D Beam Envelope Equations: An Ultrafast Computational Approach for Interactive Modeling of Accelerator Structures

By:

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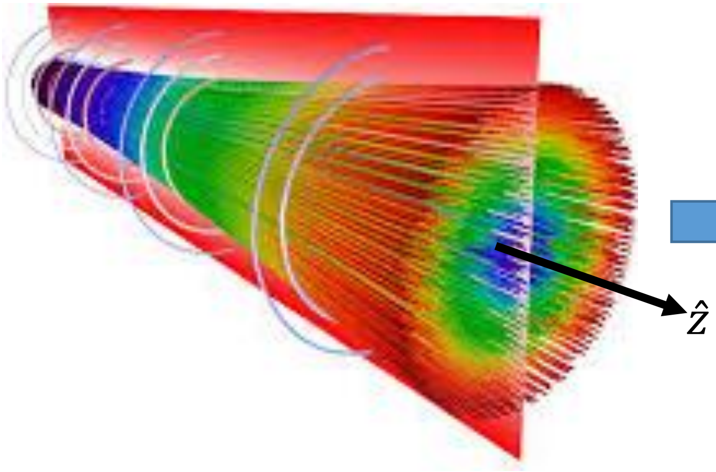
1.1 Equations of Motion



$$\vec{p} = \gamma\vec{\beta}$$

$$\frac{d\vec{r}}{dt} = \vec{\beta}c \quad , \eta = q/mc^2$$

$$\frac{d\vec{p}}{dt} = \eta\left(\vec{E} + \vec{\beta}c \times \vec{B}\right)$$

EQM

$$\gamma = \sqrt{1 + p^2}$$

$$\vec{\beta} = \frac{\vec{p}}{\sqrt{1 + p^2}}$$

$$\frac{d\vec{r}}{dt} = \frac{\vec{p}c}{\sqrt{1 + p^2}}$$

$$\frac{d\vec{p}}{dt} = \eta\left(\vec{E} + \frac{\vec{p} \times c\vec{B}}{\sqrt{1 + p^2}}\right)$$

Self-Consistent EQM

$\vec{E}$ & $\vec{B}$

1.1 Equations of Motion

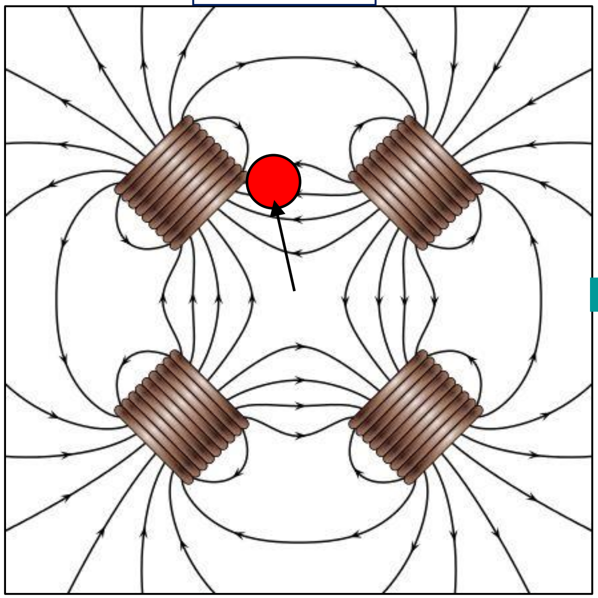
Ex Fields

$$\vec{E} = \vec{E}^e + \vec{E}^s$$

$$\vec{B} = \vec{B}^e + \vec{B}^s$$

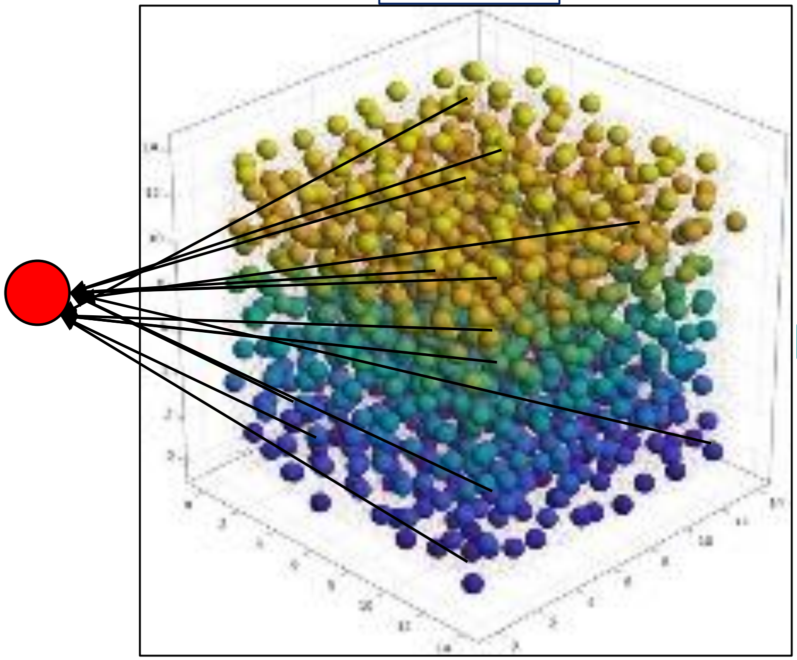
Sp Fields

Ex Fields



Costing-Time $\propto N_p$

Sp Fields



Costing-Time $\propto {N_p}^2$

A bunch of $10k$ Macro-Particles at each time step requires for

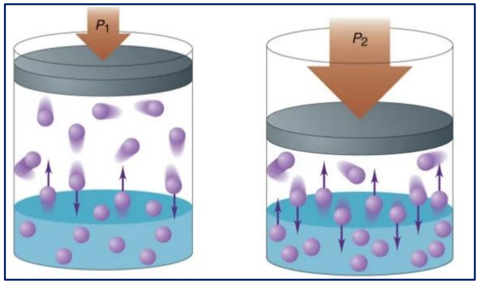
Calculation of $200M$ EM-fields

+

$30k$ Coupled Differential Equation

1.1 Equations of Motion

Gas



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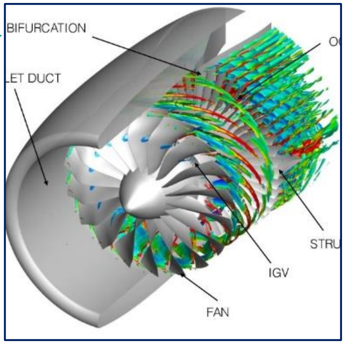
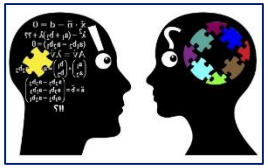
A Many-Particle System

Hydro-Dynamics Consideration

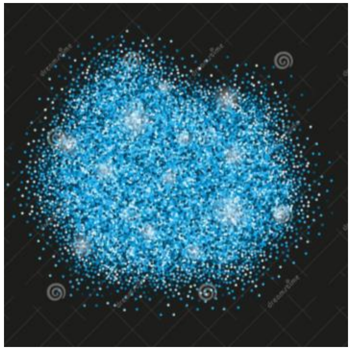


$$pV = Nk_BT$$

Relates all general parameters of the gas totally independent of the gas detailed structure



Bunch



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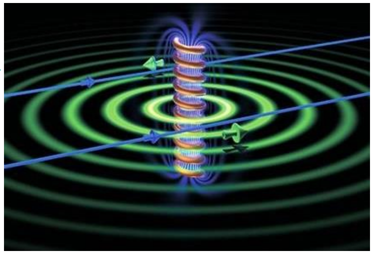
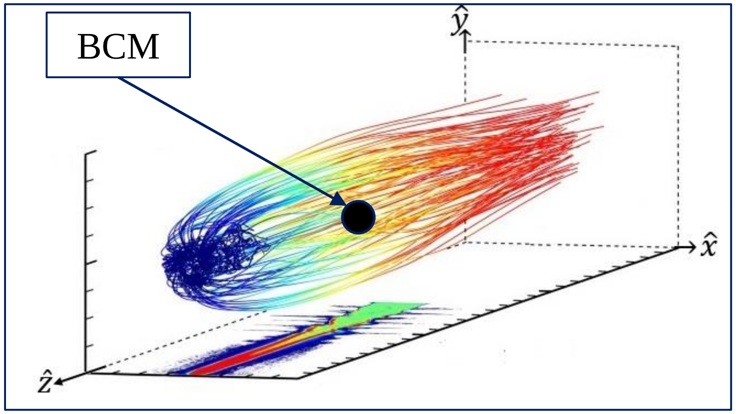
A Many-Particle System

Electro-Dynamics



1.2 Ray Equations



BCM is not affected by the space-charge forces

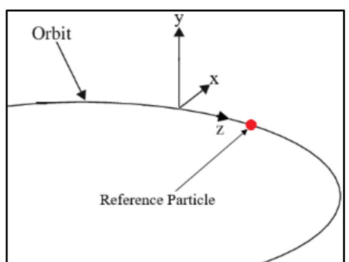
Definition of an artificial reference particle located always at the bunch center of mass $\vec{r}_0 = z_0\hat{z}$

Relativistic Factor: γ_0

Normalized Velocity: $\beta_0\hat{z}$

Normalized Momentum: $\overrightarrow{p_0} = \gamma_0\beta_0\hat{z}$

$t_0 \rightarrow \vec{r}_0 = z_0\hat{z}$



$$\frac{d(mc^2\gamma_0)}{dt_0} = \mathrm{q}\left(\vec{E}(\vec{r}_0) + \vec{\beta}_0 c \times \vec{B}(\vec{r}_0)\right)\cdot\vec{\beta}_0 c$$

$$\rightarrow \frac{d\gamma_0}{dt_0} = \eta c E_z(\vec{r}_0)\beta_0$$

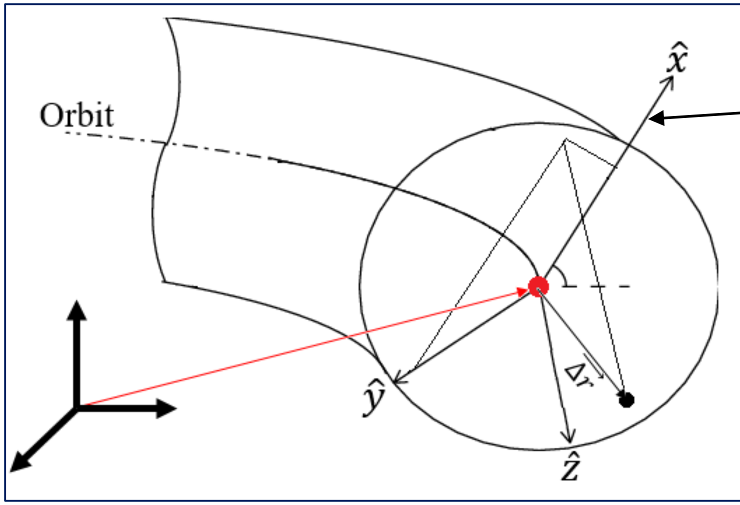
$$\frac{d}{dt_0} = \frac{dz_0}{dt_0}\frac{d}{dz_0} = \beta_0 c\frac{d}{dz_0}$$

Reference Particle SEQM

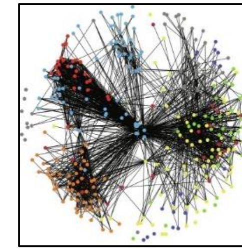
$$\frac{d\gamma_0}{dz_0} = \eta E_z(z_0\hat{z})$$

$$\frac{dt_0}{dz_0} = \frac{\gamma_0}{c\sqrt{{\gamma_0}^2 - 1}}$$

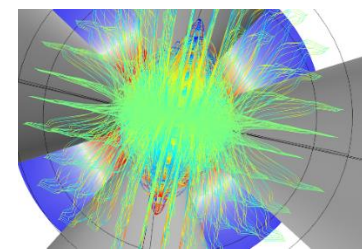
1.2 Ray Equations



Moving Frame



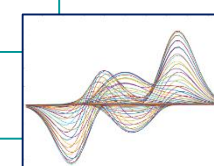
We report all the particle coordinates with respect to the moving frame by a 6D vector $[\Delta x, \Delta y, \Delta z, p_x, p_y, p_z]$,



Ray Equations

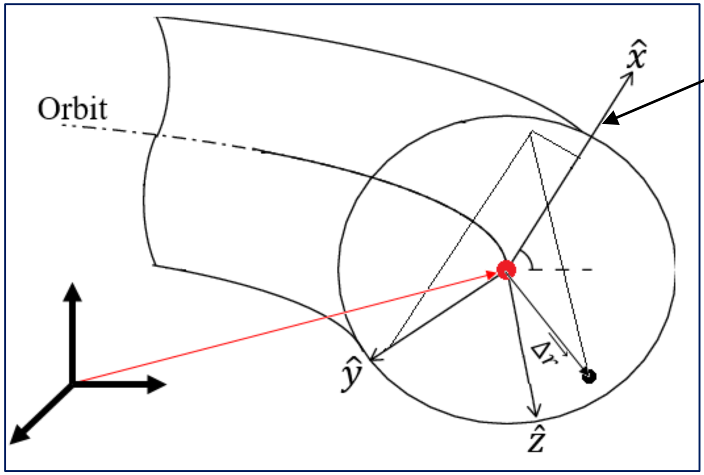
EQM + $\frac{d}{dt_0} = \beta_0 c \frac{d}{dz_0}$

$$\gamma' = \frac{\eta \vec{E} \cdot \vec{\beta}}{\beta_0}, \ \beta_u = \beta_0(\hat{u} \,.\, \hat{z} + \Delta u')$$



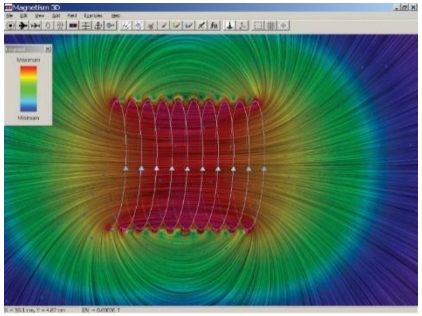
$$\Delta u'' + \frac{{\gamma_0}'}{{p_0}^3}\beta_u = \frac{\eta}{\gamma {\beta_0}^2}\left[\left(\vec{E} + \vec{\beta} \times c\vec{B}\right)_u - (\vec{E} \cdot \vec{\beta})\beta_u\right]$$

1.2 Ray Equations



Moving Frame

Solenoidal Magnet



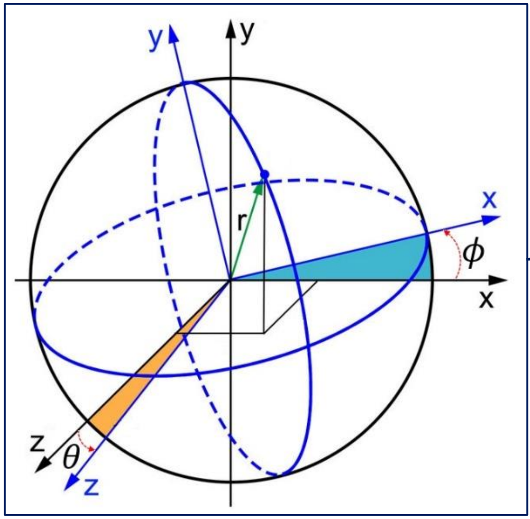
$$\overrightarrow{B^{sm}} = \left(\hat{z} - \hat{x}\frac{\Delta x}{2}\frac{d}{dz} - \hat{y}\frac{\Delta y}{2}\frac{d}{dz}\right)\mu^{sm}(z)$$

$$\mu^{sm} = \left(\vec{B}^{sm} \cdot \hat{z}\right)\Big|^{\Delta\vec{r}=0}$$

$$\begin{aligned}
&(\Delta x + i\Delta y)'' \\
&+ \left(\frac{\gamma_0\gamma_0{}'}{\gamma_0{}^2 p_0{}^2} + \frac{\eta\left(\vec{E}\cdot\vec{\beta}\right)}{\gamma\beta_0} + \frac{i\eta c}{\gamma\beta_0}\mu^{sm}\right)(\Delta x + i\Delta y)' \\
&+ \frac{i\eta c\beta_z}{2\gamma\beta_0{}^2}\frac{d\mu^{sm}}{dz}(\Delta x + i\Delta y) \\
&= \frac{\eta}{\gamma\beta_0{}^2}\left(E_x + iE_y\right) \\
&+ \frac{i\eta c\beta_z}{\gamma\beta_0{}^2}\left((\mathrm{B}_x - B_x^{sm}) + i\left(\mathrm{B}_y - B_y^{sm}\right)\right)
\end{aligned}$$

Coupling Due to Solenoidal Fields

1.2 Ray Equations



$$\theta = 0 \quad \rightarrow \quad [\Delta z]_{prev} = [\Delta z]_{new}$$

$$[\Delta x + i\Delta y]_{prev} = [\Delta x + i\Delta y]_{new} e^{i\phi}$$

$$\phi' = -\frac{\eta c}{2\gamma\beta_0}\mu^{sm}$$

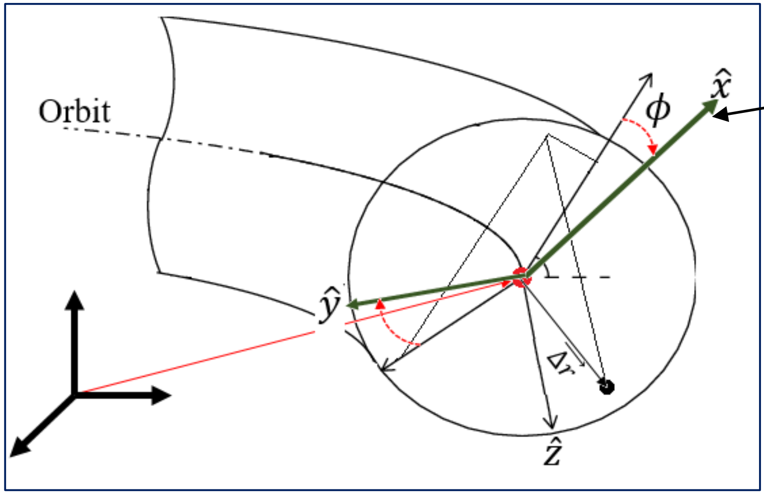
$$(\Delta x + i\Delta y)'' + \left(\frac{\gamma_0\gamma_0'}{\gamma_0{}^2 p_0{}^2} + \frac{\eta(\vec{E}\cdot\vec{\beta})}{\gamma\beta_0}\right)(\Delta x + i\Delta y)'$$

$$= +\frac{\eta}{\gamma\beta_0{}^2}(E_x + iE_y) - \left(\frac{\eta c}{2\gamma\beta_0}\mu^{sm}\right)^2(\Delta x + i\Delta y)$$

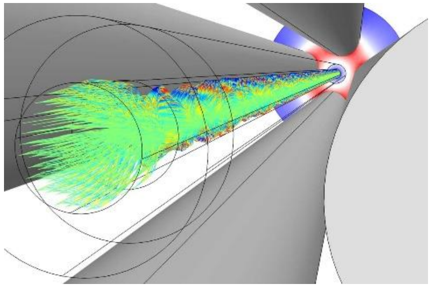
$$+\frac{i\eta c\beta_z}{\gamma\beta_0{}^2}\left((\mathrm{B}_x - B_x^{sm}) + i(\mathrm{B}_y - B_y^{sm})\right)$$

No coupling at least up to the first order

1.2 Ray Equations



Larmor Moving Frame



Ray Equations in Larmor Moving Frame

$$\gamma' = \frac{\eta \vec{E}\cdot\vec{\beta}}{\beta_0},\ \ \beta_u = \beta_0(\hat{u}\,.\,\hat{z} +\ \Delta u')$$

$$\Delta u'' + \frac{{\gamma_0}'}{{p_0}^3}\beta_u$$

$$= -\frac{\eta\left(\vec{E}\cdot\vec{\beta}\right)\beta_u}{\gamma{\beta_0}^2} - \frac{(1-\hat{u}\cdot\hat{z})\eta^2c^2\left(\overrightarrow{B^{sm}}\cdot\hat{z}\right)^2\Delta u}{4\gamma^2{\beta_0}^2}$$

$$+\ \frac{\eta\left(\vec{E}+\vec{\beta}c\times\vec{B}-(1-\hat{u}\cdot\hat{z})\vec{\beta}c\times\overrightarrow{B^{sm}}\right)\cdot\hat{u}}{\gamma{\beta_0}^2}$$

solenoidal magnetic field

2.1 6D Envelope Equations

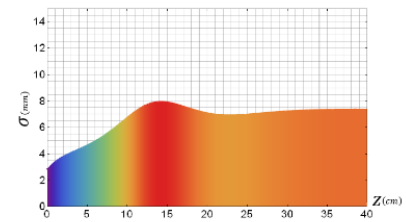
Ray Equations $\times$ Δu $+$ Averaging Over Whole Bunch $+$

$$\sigma_u = \sqrt{\langle \Delta u^2 \rangle}$$

$$\Lambda_u = \sqrt{< \Delta u'^2 >}$$

Envelope Equations

$$\sigma_u'' + \frac{\gamma_0 \gamma_0'}{\gamma_0^{\,2} p_0^{\,2}} \sigma_u' = F_u^e + F_u^s + F_u^\varepsilon$$



$$F_u^e = -\frac{\eta}{{\beta_0}^2 \sigma_u} \langle \frac{\left(\vec{E^e} . \vec{\beta}\right) \beta_u \Delta u}{\gamma} \rangle - \frac{\eta^2 c^2 (1 - \hat{u} \cdot \hat{z})}{4 {\beta_0}^2 \sigma_u} \langle \frac{\left(\vec{B^{sm}} . \hat{z}\right)^2 \Delta u^2}{\gamma^2} \rangle + \frac{\eta}{{\beta_0}^2 \sigma_u} \langle \frac{\left(\vec{E^e} + \vec{\beta} c \times \vec{B^e} - (1 - \hat{u} \cdot \hat{z}) \vec{\beta} c \times \vec{B^{sm}}\right) \cdot \Delta u \hat{u}}{\gamma} \rangle$$

Ex F-Type Force

$$F_u^s = -\frac{\eta}{{\beta_0}^2 \sigma_u} \langle \frac{\left(\vec{E^s} \cdot \vec{\beta}\right) \beta_u \Delta u}{\gamma} \rangle + \frac{\eta}{{\beta_0}^2 \sigma_u} \langle \frac{\left(\vec{E^s} + \vec{\beta} c \times \vec{B^s}\right) \cdot \Delta u \hat{u}}{\gamma} \rangle$$

Sp F-Type Force

$$F_u^\varepsilon = \frac{{\Lambda_u}^2 - {\sigma_u'}^2}{\sigma_u}, \Lambda_u = \sqrt{< \Delta u'^2 >}$$

Emit F-type Force

$\vec{\sigma}$ ✔

$\vec{\Lambda}$ ✖

2.1 6D Envelope Equations

${\Lambda_u}^2$

$$\frac{d\langle {\Delta u'}^2\rangle}{dz_0}$$

+

Ray Equations in Larmor Moving Frame

$$\frac{d\Lambda_u}{dz_0} + \frac{\gamma_0{\gamma_0}'}{{\gamma_0}^2{p_0}^2}\Lambda_u = G_u^e + G_u^s$$

$$G_u^e = -\frac{\eta}{{\beta_0}^2\Lambda_u}\left\langle\frac{\left(\vec{E^e}\cdot\vec{\beta}\right)\beta_u\Delta u'}{\gamma}\right\rangle - \frac{\eta^2c^2(1-\hat{u}\cdot\hat{z})}{4{\beta_0}^2\Lambda_u}\left\langle\frac{\left(\vec{B^{sm}}\cdot\hat{z}\right)^2\Delta u\Delta u'}{\gamma^2}\right\rangle + \frac{\eta}{{\beta_0}^2\Lambda_u}\left\langle\frac{\left(\vec{E^e}+\vec{\beta}c\times\vec{B^e}-(1-\hat{u}\cdot\hat{z})\vec{\beta}c\times\vec{B^{sm}}\right)\cdot\Delta u'\hat{u}}{\gamma}\right\rangle$$

Ex G-type Force

6D Integrals

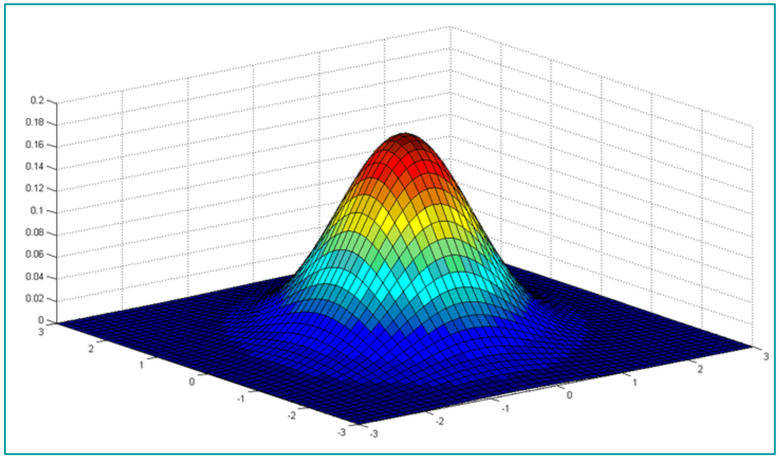
$$G_u^s = -\frac{\eta}{{\beta_0}^2\Lambda_u}\left\langle\frac{\left(\vec{E^s}\cdot\vec{\beta}\right)\beta_u\Delta u'}{\gamma}\right\rangle + \frac{\eta}{{\beta_0}^2\Lambda_u}\left\langle\frac{\left(\vec{E^s}+\vec{\beta}c\times\vec{B^s}\right)\cdot\Delta u'\hat{u}}{\gamma}\right\rangle$$

Sp G-type Force

2.3 Bunch Distribution

Dynamics independent of the bunch detailed structure

No matter what is the distribution just it must provide all 6 parameters



$$\mathcal{F}_u(\Delta u,\Delta u') = \frac{p_0}{2\varepsilon_u}\, e^{-\left[\left(\frac{p_0\Lambda_u}{\sqrt{2}\varepsilon_u}\Delta u\right)^2 - 2\frac{{p_0}^2\sigma_u\sigma_u{}'}{2{\varepsilon_u}^2}\Delta u\Delta u' + \left(\frac{p_0\Lambda_u}{\sqrt{2}\varepsilon_u}\Delta u'\right)^2\right]}$$

$$\sqrt{\langle \Delta u^2\rangle} = \sigma_u$$

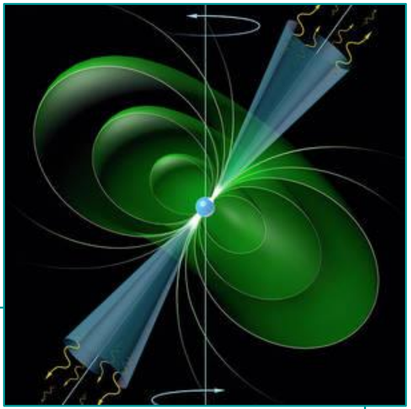
$$\sqrt{\langle \Delta u'^2\rangle} = \Lambda_u$$

$$p_0\sqrt{\langle \Delta u^2\rangle\langle \Delta u'^2\rangle - \langle \Delta u\Delta u'\rangle^2} = \varepsilon_u$$

6D Gaussian distribution

$$\mathcal{F}(\Delta x,\Delta y,\Delta z,\Delta x',\Delta y',\Delta z') = \mathcal{F}_x(\Delta x,\Delta x')\mathcal{F}_y(\Delta y,\Delta y')\mathcal{F}_z(\Delta z,\Delta z')$$

3.1 Space-Charge Forces



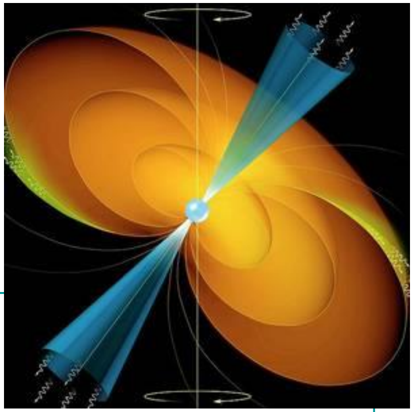
Space Charge Forces

$$f_b = \frac{\eta q_b}{8\pi\sqrt{\pi}\epsilon_0}$$

$$\begin{aligned}
F_x^s \cong &+ \frac{f_b}{{\beta_0}^2{\gamma_0}^3} \times \frac{\alpha_x}{\sigma_x\sigma_z} \\
&- \frac{f_b}{\gamma_0}\left\{\frac{\left({\Lambda_x}^2 + 2{\sigma_x'}^2\right)\alpha_x - {\sigma_x'}^2\alpha_{xx}}{2\sigma_x\sigma_z} + \frac{8{\Lambda_y}^2\alpha_x - {\sigma_y'}^2\alpha_{xy}}{16\sigma_x\sigma_z} + \frac{\left(1-{p_0}^2\right)\left(2{\Lambda_z}^2\alpha_x - {\sigma_z'}^2\alpha_{xz}\right)}{4\sigma_x\sigma_z}\right\} \\
&- \frac{f_b}{\gamma_0}\left\{\frac{\left({\Lambda_x}^2 + 2{\sigma_x'}^2\right)\alpha_x - {\sigma_x'}^2\alpha_{xx}}{\sigma_x\sigma_z} + \frac{\sigma_x'\sigma_y'\left(8\alpha_y - \alpha_{xy}\right)}{8\sigma_z\sigma_y} + \frac{\left(1-{p_0}^2\right)\sigma_x'\sigma_z'\left(2{\gamma_0}^2{\sigma_z}^2\alpha_z - \sigma_x\sigma_y\alpha_{xz}\right)}{2\sigma_x\sigma_y{\gamma_0}^2{\sigma_z}^2}\right\}
\end{aligned}$$

$$\begin{aligned}
G_x^s = &+ \frac{f_b\sigma_x'}{{\beta_0}^2{\gamma_0}^3\Lambda_x}\frac{\alpha_x}{\sigma_x\sigma_z} \\
&- \frac{f_b\sigma_x'}{2\gamma_0\Lambda_x}\left\{3\,\frac{3{\Lambda_x}^2\alpha_x - {\sigma_x'}^2\alpha_{xx}}{\sigma_x\sigma_z} + \frac{8{\Lambda_y}^2\alpha_x - {\sigma_y'}^2\alpha_{xy}}{8\sigma_x\sigma_z} + \left(1-{p_0}^2\right)\frac{2{\Lambda_z}^2\alpha_x - {\sigma_z'}^2\alpha_{xz}}{2\sigma_x\sigma_z}\right\} \\
&- \frac{f_b\sigma_y'}{8\gamma_0\Lambda_x}\left\{\frac{8{\Lambda_x}^2\alpha_y - {\sigma_x'}^2\alpha_{xy}}{\sigma_y\sigma_z}\right\} - \frac{f_b\sigma_z'}{2\gamma_0\Lambda_x}\left\{\left(1-{p_0}^2\right)\frac{2{\gamma_0}^2{\sigma_z}^2{\Lambda_x}^2\alpha_z - {\sigma_x'}^2\sigma_x\sigma_y\alpha_{xz}}{\sigma_x\sigma_y{\gamma_0}^2{\sigma_z}^2}\right\}
\end{aligned}$$

3.1 Space-Charge Forces



Space Charge Forces

$$f_b = \frac{\eta q_b}{8\pi\sqrt{\pi}\epsilon_0}$$

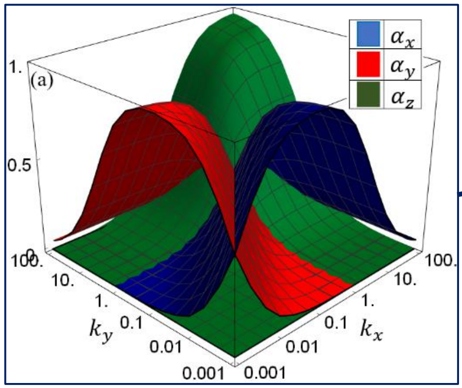
$$F_z^s \cong +\frac{f_b}{{\beta_0}^2{\gamma_0}^3} \times \frac{\alpha_z}{\sigma_x\sigma_y}$$
$$-\frac{f_b}{\gamma_0}\left\{\frac{3(1-{p_0}^2)\left(2\left({\Lambda_z}^2+2{\sigma_z'}^2\right)\alpha_z - {\sigma_z'}^2\alpha_{zz}\right)}{4\sigma_x\sigma_y} + \frac{2{\gamma_0}^2{\sigma_z}^2{\Lambda_x}^2\alpha_z - {\sigma_x'}^2\sigma_x\sigma_y\alpha_{xz}}{4\sigma_x\sigma_y{\gamma_0}^2{\sigma_z}^2}\right.$$

$$G_z^s = +\frac{f_b\sigma_z'}{{\beta_0}^2{\gamma_0}^3\Lambda_z} \times \frac{\alpha_z}{\sigma_x\sigma_y}$$
$$-\frac{f_b\sigma_z'}{2\gamma_0\Lambda_z}\left\{3(1-{p_0}^2)\frac{6{\Lambda_z}^2\alpha_z - {\sigma_z'}^2\alpha_{zz}}{2\sigma_x\sigma_y} + \frac{2{\gamma_0}^2{\sigma_z}^2{\Lambda_x}^2\alpha_z - {\sigma_x'}^2\sigma_x\sigma_y\alpha_{xz}}{2\sigma_x\sigma_y{\gamma_0}^2{\sigma_z}^2} + \frac{2{\gamma_0}^2{\sigma_z}^2{\Lambda_y}^2\alpha_z - {\sigma_y'}^2\sigma_x\sigma_y\alpha_{yz}}{2\sigma_x\sigma_y{\gamma_0}^2{\sigma_z}^2}\right\}$$
$$-\frac{f_b\sigma_x'}{2\gamma_0\Lambda_z}\left\{\frac{2{\Lambda_z}^2\alpha_x - {\sigma_z'}^2\alpha_{xz}}{\sigma_x\sigma_z}\right\} - \frac{f_b\sigma_y'}{2\gamma_0\Lambda_z}\left\{\frac{2{\Lambda_z}^2\alpha_y - {\sigma_z'}^2\alpha_{yz}}{\sigma_y\sigma_z}\right\}$$

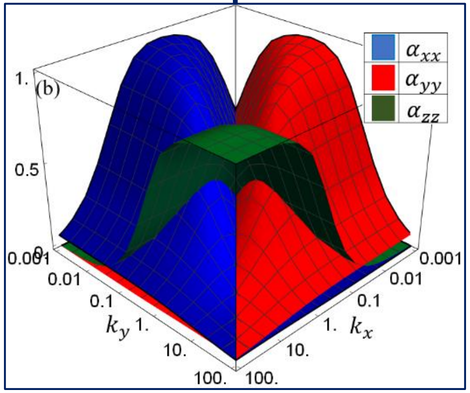
3.1 Space-Charge Forces

1st order	$\{\alpha_x, \alpha_x, \alpha_x\}$
2nd Order	$\{\alpha_{xx}, \alpha_{yy}, \alpha_{zz}\}$
3rd Order	$\{\alpha_{xy}, \alpha_{xz}, \alpha_{yz}\}$

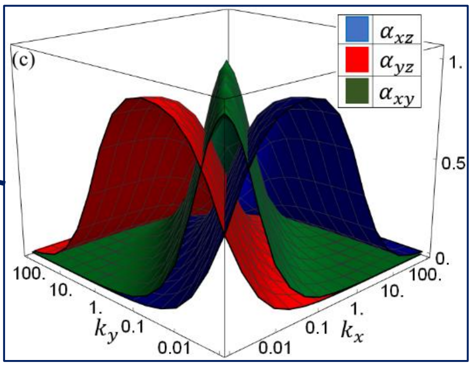
$$k_u = \frac{\sigma_u}{\gamma_0 \sigma_z}$$



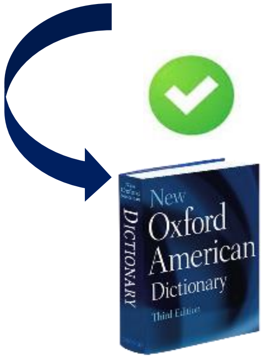
$$\alpha_x = \int_0^{+\infty} \frac{{k_x}^2 s ds}{\sqrt{(s^2 + {k_x}^2)^3 (s^2 + {k_y}^2)(s^2 + 1)}}$$



$$\alpha_{xx} = \int_0^{+\infty} \frac{3{k_x}^4 s ds}{2\sqrt{(s^2 + {k_x}^2)^5 (s^2 + {k_y}^2)(s^2 + 1)}}$$

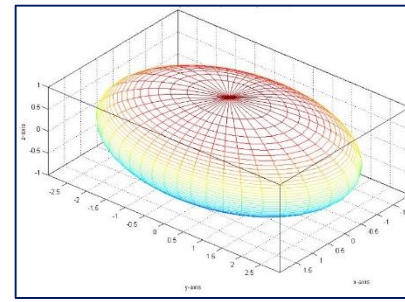


$$\alpha_{xz} = \int_0^{+\infty} \frac{{k_x}^2 s ds}{\sqrt{(s^2 + {k_x}^2)^3 (s^2 + {k_y}^2)(s^2 + 1)^3}}$$

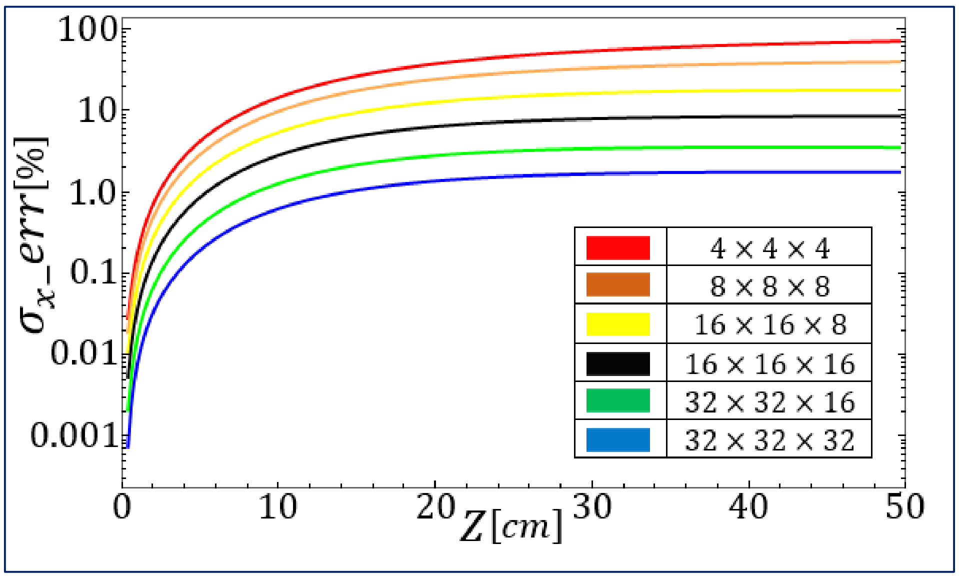


3.1 Space-Charge Forces

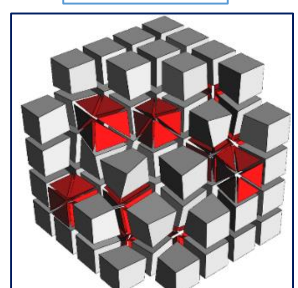
Specifications	$E_k[MeV]$	$\sigma_E[\%]$	$\sigma_x[mm]$	$\sigma_y[mm]$	$\sigma_z[mm]$	$\varepsilon_{nx}[\mu m]$	$\varepsilon_{ny}[\mu m]$	$\varepsilon_{nz}[\mu m]$
Value	5	1	$\sqrt{2}$	$2\sqrt{2}$	0.030	0.05	0.10	2.93



Astra Vs Exact Approach



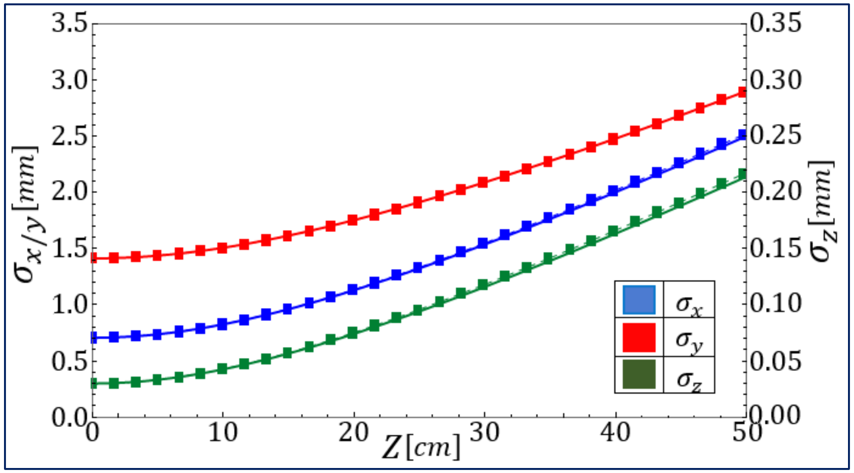
ASTRA



$2^n \times 2^m \times 2^l$

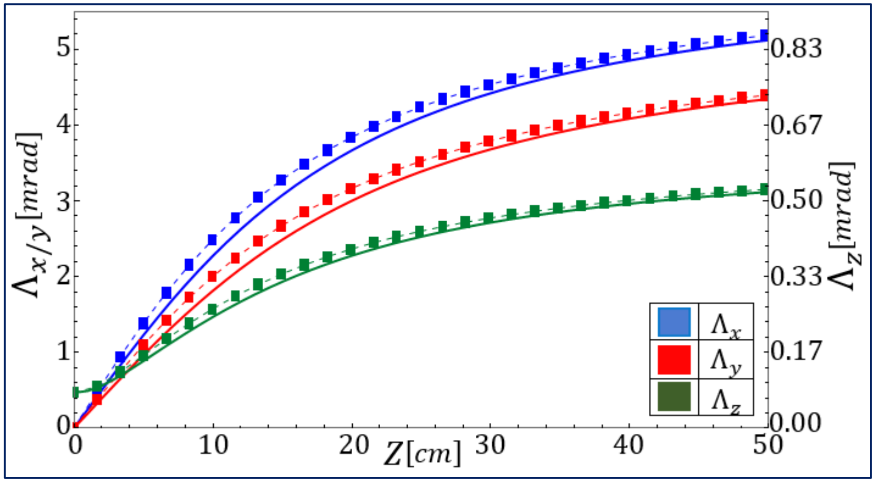
$2^5 \times 2^5 \times 2^5$
Simulation-Time :
$> 6h$

3.1 Space-Charge Forces



Simulation-Time : $> 6h$

Simulation-Time : $< 1s$

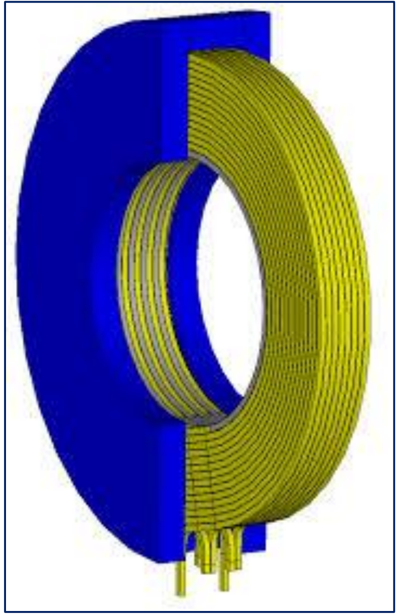


3.2 Solenoidal Magnet Forces

$$F_x^e = -\frac{\eta^2 c^2}{4{p_0}^2}\left(\mu^{sm2} + \mu_z^{sm2}{\sigma_z}^2\right)\sigma_x + \eta^2 c^2 \mu^{sm}\mu_z^{sm}\sigma_x\sigma_z{\sigma_z}' + \frac{\eta^2 c^2 \mu^{sm2}}{4}\sigma_x\left(2{\sigma_x}'^2 + {\Lambda_x}^2 + {\Lambda_y}^2 + {\Lambda_z}^2\right) + \frac{\eta^2 c^2 \mu_z^{sm2}}{4}\sigma_x{\sigma_z}^2\left(2{\sigma_x}'^2 + 2{\sigma_z}'^2 + {\Lambda_x}^2 + {\Lambda_y}^2 + {\Lambda_z}^2\right)$$

$$F_z^e = 0$$

$$\mu^{sm} = \left(\vec{B}^{sm}\cdot\hat{z}\right)|^{\Delta\vec{r}=0}$$

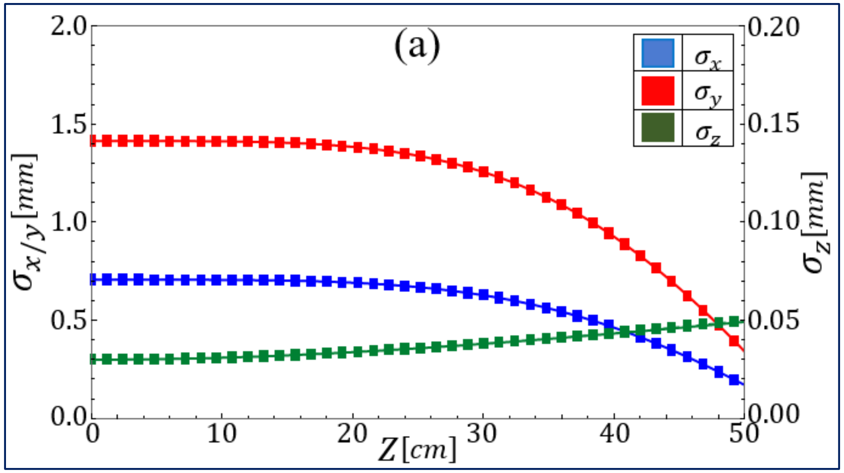


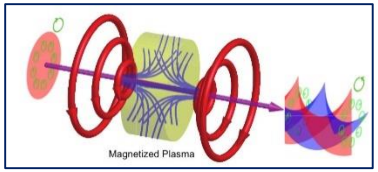
$$\mu_z^{sm} = \left(\frac{\partial}{\partial z}\vec{B}^{sm}\cdot\hat{z}\right)|^{\Delta\vec{r}=0}$$

$$G_x^e = -\frac{\eta^2 c^2}{4{p_0}^2\Lambda_x}\left(\mu^{sm2} + \mu_z^{sm2}{\sigma_z}^2\right)\sigma_x{\sigma_x}' + \frac{\eta^2 c^2 \mu^{sm}\mu_z^{sm}}{\Lambda_x}\sigma_x{\sigma_x}'\sigma_z{\sigma_z}' + \frac{\eta^2 c^2 \mu^{sm2}}{4\Lambda_x}\sigma_x{\sigma_x}'\left(3{\Lambda_x}^2 - {\Lambda_y}^2 - {\Lambda_z}^2\right) + \frac{\eta^2 c^2 \mu_z^{sm2}}{4\Lambda_x}\sigma_x{\sigma_x}'{\sigma_z}^2\left(2{\sigma_z}'^2 + 3{\Lambda_x}^2 + {\Lambda_y}^2 + {\Lambda_z}^2\right)$$

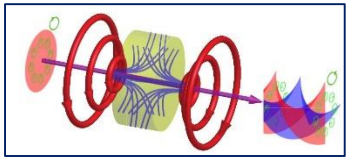
$$G_z^e = 0$$

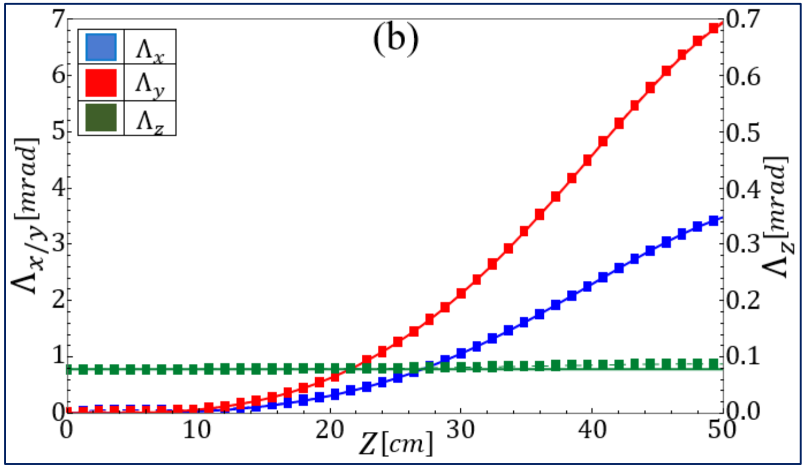
3.2 Solenoidal Magnet Forces





$$\mu^{sm}(z) = 0.5\, z$$





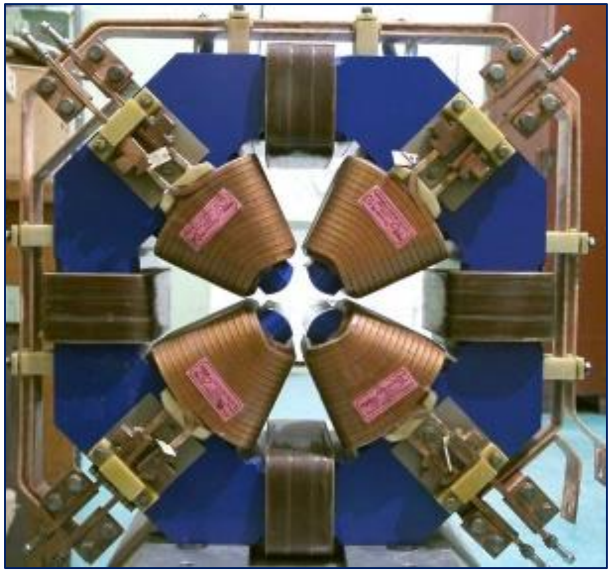
3.3 Quadrupole Magnet Forces

$$F_x^e = -k^{qm}\sigma_x + \frac{k^{qm}{p_0}^2}{2}\sigma_x\{(2+{\gamma_0}^2){\Lambda_z}^2 + 2{\sigma_x}'^2 + {\Lambda_x}^2 + {\Lambda_y}^2\}$$

$$F_y^e = +k^{qm}\sigma_y - \frac{k^{qm}{p_0}^2}{2}\sigma_y\{(2+{\gamma_0}^2){\Lambda_z}^2 + 2{\sigma_y}'^2 + {\Lambda_y}^2 + {\Lambda_x}^2\}$$

$$F_z^e = -k^{qm}{p_0}^2{\sigma_z}'(\sigma_x{\sigma_x}' - \sigma_y{\sigma_y}')$$

$$\overrightarrow{B^{qm}} = \frac{p_0 k^{qm}}{\eta c}(\Delta y\hat{x} + \Delta x\hat{y})$$

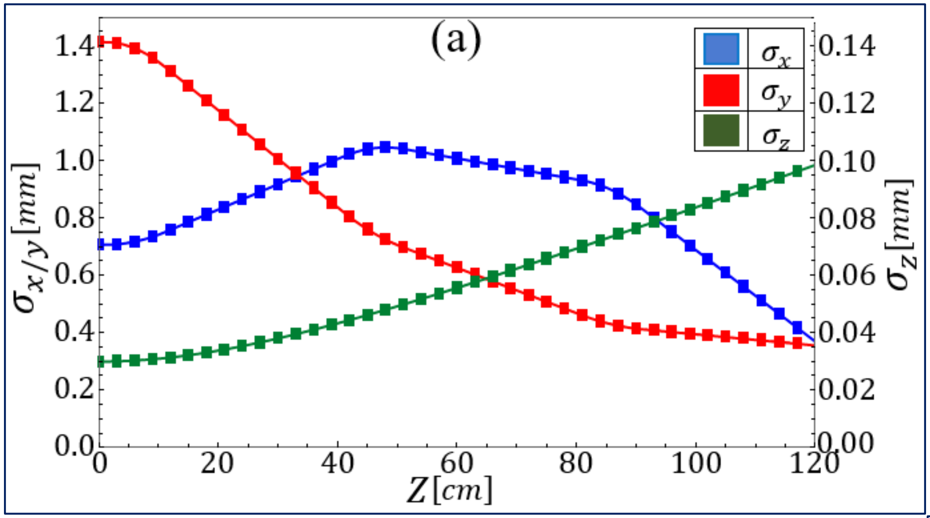


$$G_x^e = -\frac{k^{qm}}{\Lambda_x}\sigma_x{\sigma_x}' + \frac{k^{qm}{p_0}^2}{2\Lambda_x}\sigma_x{\sigma_x}'\{(2+{\gamma_0}^2){\Lambda_z}^2 + 3{\Lambda_x}^2 + {\Lambda_y}^2\}$$

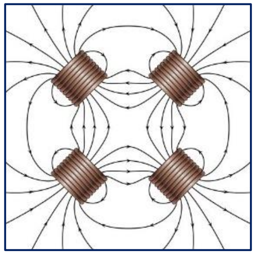
$$G_y^e = +\frac{k^{qm}}{\Lambda_y}\sigma_y{\sigma_y}' - \frac{k^{qm}{p_0}^2}{2\Lambda_y}\sigma_y{\sigma_y}'\{(2+{\gamma_0}^2){\Lambda_z}^2 + 3{\Lambda_y}^2 + {\Lambda_x}^2\}$$

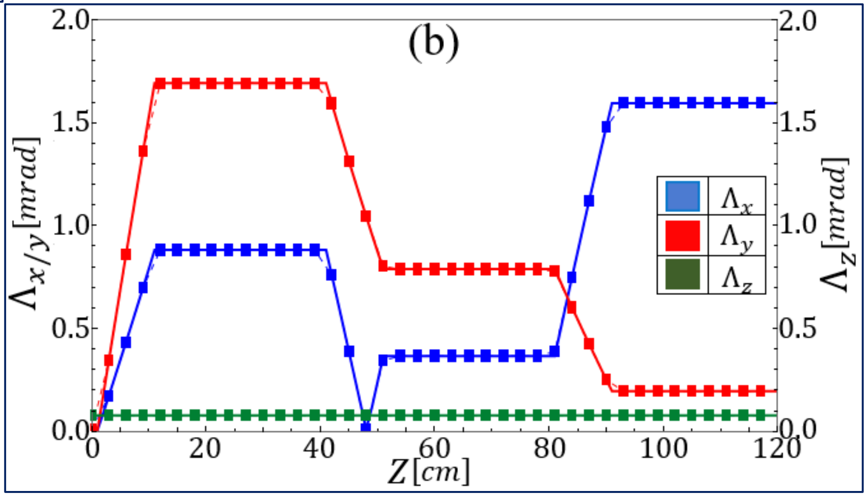
$$G_z^e = -k^{qm}{p_0}^2\Lambda_z(\sigma_x{\sigma_x}' - \sigma_y{\sigma_y}')$$

3.3 Quadrupole Magnet Forces



$Element$	$k^{qm}[1/m^2]$	$L[cm]$	$z[cm]$
Q_1	-12.221	10	6
Q_2	$+12.018$	10	46
Q_3	$+13.776$	10	86





3.4 Electrostatic and RF Forces

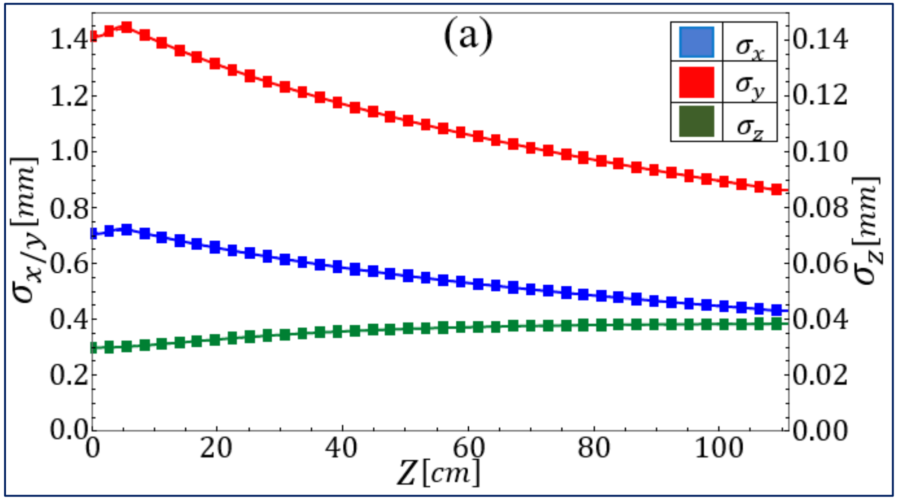
$$F_x^e = -\frac{\eta\left(\mathcal{E}_z^{rf} + \beta_0\mathcal{E}_t^{rf}\right)}{2\gamma_0{\beta_0}^2}\sigma_x - \frac{\eta\mathcal{E}^{rf}}{\gamma_0}{\sigma_x}'$$
$$+\frac{\eta\mathcal{E}_z^{rf}}{2\gamma_0}{\sigma_x}'\{\sigma_y{\sigma_y}' - 2(1-{p_0}^2)\sigma_z{\sigma_z}'\} + \frac{\eta\mathcal{E}_z^{rf}}{4\gamma_0}\sigma_x\{{\gamma_0}^4{\Lambda_z}^2 + 2(2+{\gamma_0}^2){\sigma_x}'^2 + (2+{\gamma_0}^2){\Lambda_x}^2 + {\gamma_0}^2{\Lambda_y}^2\}$$
$$+\frac{\eta p_0\mathcal{E}_t^{rf}}{4}\sigma_x\{(2+{\gamma_0}^2){\Lambda_z}^2 + 2{\sigma_x}'^2 + \left({\Lambda_x}^2 + {\Lambda_y}^2\right)\} + \frac{\eta {p_0}^2\mathcal{E}^{rf}}{2\gamma_0}{\sigma_x}'\{(2+{\gamma_0}^2){\Lambda_z}^2 + 3{\Lambda_x}^2 + {\Lambda_y}^2\}$$

$$F_z^e = +\frac{\eta\mathcal{E}_z^{rf}}{\gamma_0{p_0}^2}\sigma_z - \frac{3\eta\mathcal{E}^{rf}}{\gamma_0}{\sigma_z}'$$
$$+\frac{\eta(1-{p_0}^2)\mathcal{E}_z^{rf}}{2\gamma_0}{\sigma_z}'(\sigma_x{\sigma_x}' + \sigma_y{\sigma_y}') - \frac{\eta p_0\mathcal{E}_t^{rf}}{2}{\sigma_z}'(\sigma_x{\sigma_x}' + \sigma_y{\sigma_y}')$$
$$+\frac{\eta\mathcal{E}_z^{rf}}{2\gamma_0}\sigma_z\{(6-3{\gamma_0}^2-2{\gamma_0}^4)\left(2{\sigma_z}'^2 + {\Lambda_z}^2\right) - \left({\Lambda_x}^2 + {\Lambda_y}^2\right)\} + \frac{\eta{p_0}^2\mathcal{E}^{rf}}{2\gamma_0}{\sigma_z}'\{3(2+{\gamma_0}^2){\Lambda_z}^2 + \left({\Lambda_x}^2 + {\Lambda_y}^2\right)\}$$

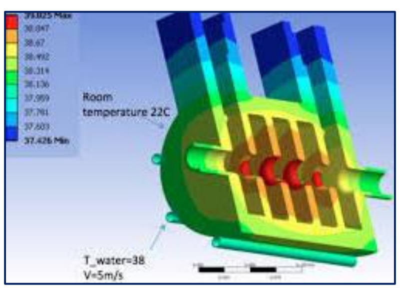
$$G_x^e = -\frac{\eta\left(\mathcal{E}_z^{rf} + \beta_0\mathcal{E}_t^{rf}\right)}{2\gamma_0{\beta_0}^2\Lambda_x}\sigma_x{\sigma_x}' - \frac{\eta\mathcal{E}^{rf}}{\gamma_0}\Lambda_x$$
$$+\frac{\eta\mathcal{E}_z^{rf}}{2\gamma_0}\Lambda_x\{3\sigma_x{\sigma_x}' + \sigma_y{\sigma_y}' - 2(1-{p_0}^2)\sigma_z{\sigma_z}'\} + \frac{\eta\gamma_0\mathcal{E}_z^{rf}}{4\Lambda_x}\sigma_x{\sigma_x}'\left({\gamma_0}^2{\Lambda_z}^2 + 3{\Lambda_x}^2 + {\Lambda_y}^2\right)$$
$$+\frac{\eta p_0\mathcal{E}_t^{rf}}{4\Lambda_x}\sigma_x{\sigma_x}'\{(3+{p_0}^2){\Lambda_z}^2 + 3{\Lambda_x}^2 + {\Lambda_y}^2\}$$

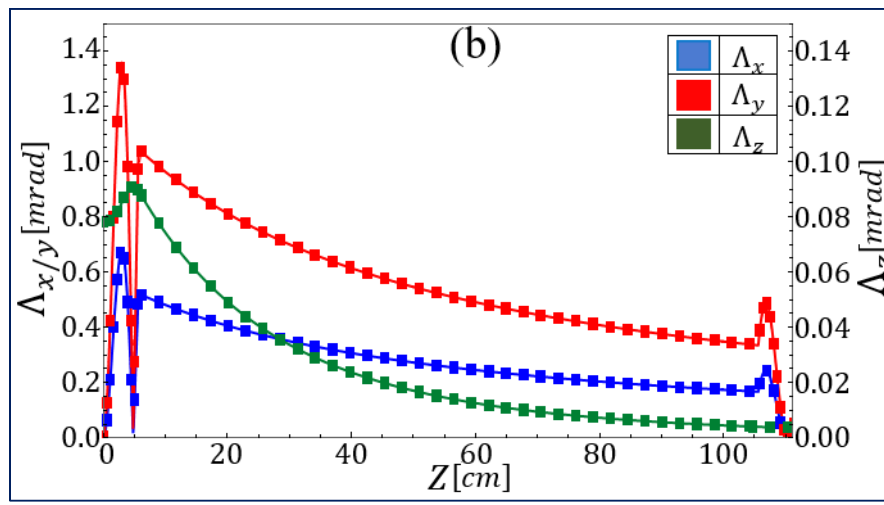
$$G_z^e = +\frac{\eta\mathcal{E}_z^{rf}}{\gamma_0{p_0}^2\Lambda_z}\sigma_z{\sigma_z}' - \frac{3\eta\mathcal{E}^{rf}}{\gamma_0}\Lambda_z$$
$$+\frac{\eta(1-{p_0}^2)\mathcal{E}_z^{rf}}{2\gamma_0}\Lambda_z\left(\sigma_x{\sigma_x}' + \sigma_y{\sigma_y}'\right) - \frac{\eta p_0\mathcal{E}_t^{rf}}{2}\Lambda_z\left(\sigma_x{\sigma_x}' + \sigma_y{\sigma_y}'\right)$$
$$+\frac{\eta\mathcal{E}_z^{rf}}{2\gamma_0\Lambda_z}\sigma_z{\sigma_z}'\{3(6-3{\gamma_0}^2-2{\gamma_0}^4){\Lambda_z}^2 - \left({\Lambda_x}^2 + {\Lambda_y}^2\right)\}$$

3.4 Electrostatic and RF Forces



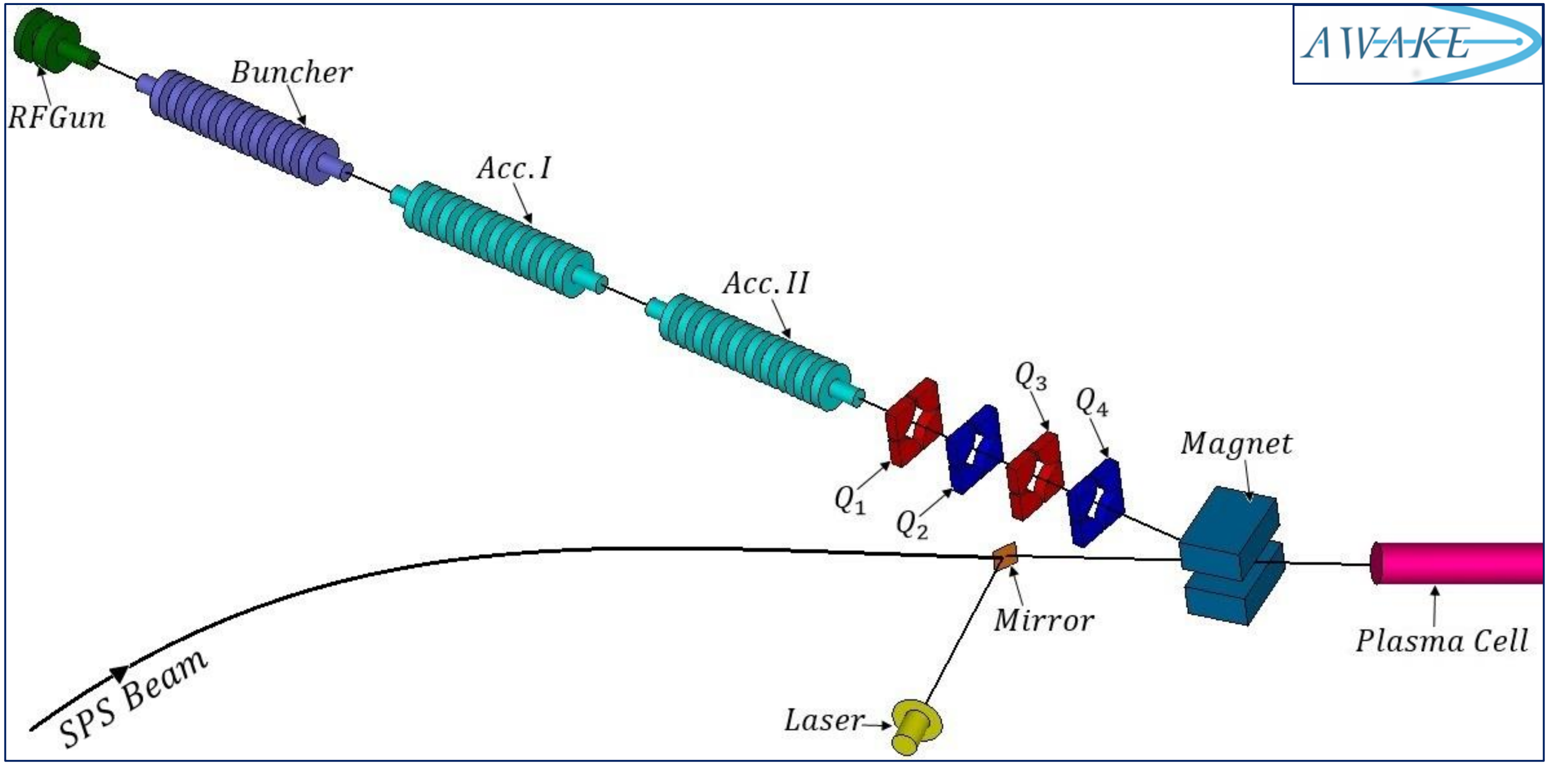
Parameter	Cell	f[GHz]	ϕ[rad]	$E_0[Mv/m]$
Value	31	3	$2\pi/3$	15



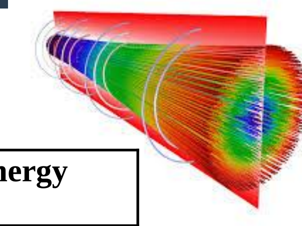


3.5 AWAKE Experiment

Injection of a compact and high-quality electron bunch at a right phase allows for a propagation over long distances with preserving emittance.



3.5 AWAKE Experiment

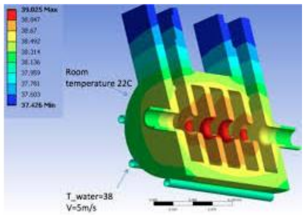


Beam Characteristics

Type	Bunch Charge	Bunch Length	Energy Spread	Emittance	Length	Energy
2nd	$\cong 200$ pC	$< 200fs$	$< 1\%$	$< 2\ \mu m$	< 5m	$\cong 200MeV$

RF Characteristics

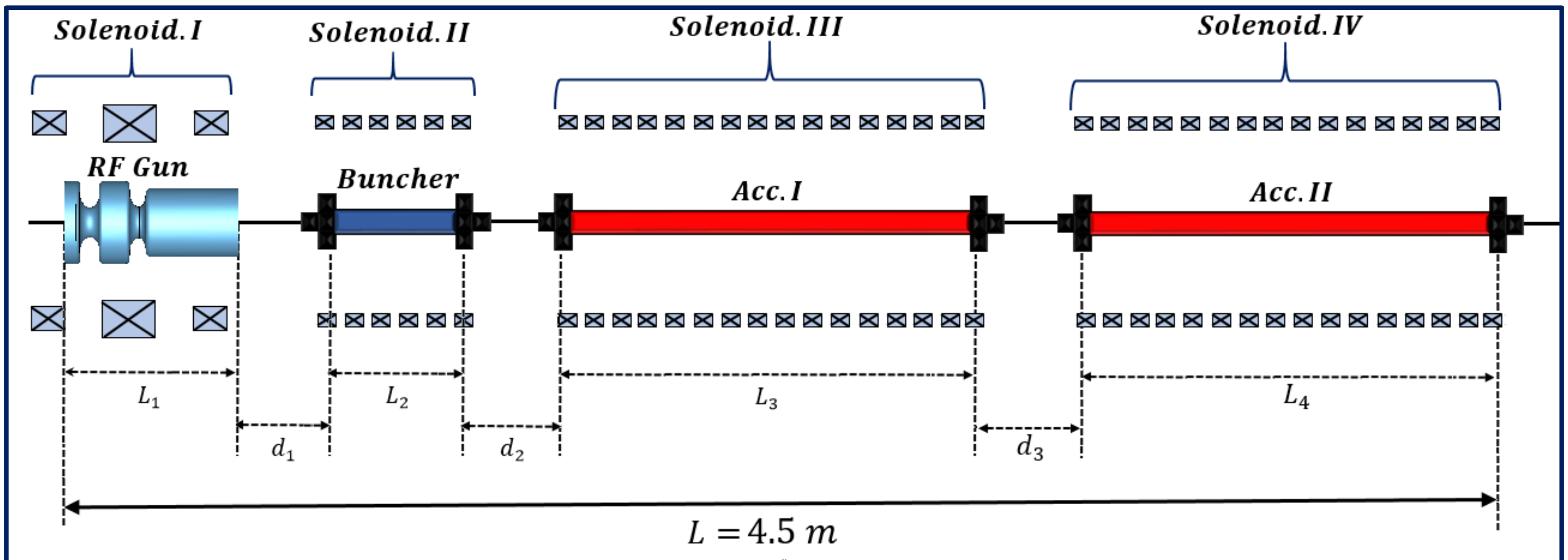
Parameter	RF Gun	Buncher	Acc. I	Acc. II
Frequency	3.0	12.0	12.0	12.0
Max Gradient	$120MV/m$	$50MV/m$	$80MV/m$	$80MV/m$
N. Cell	1.5	30	120	120



Laser Characteristics

$\lambda[nm]$	$w[ev]$	$r[mm]$	$t[ps]$	$q[pc]$
262	4.31	1.0-2.0	1.0	100-600

3.5 AWAKE Experiment



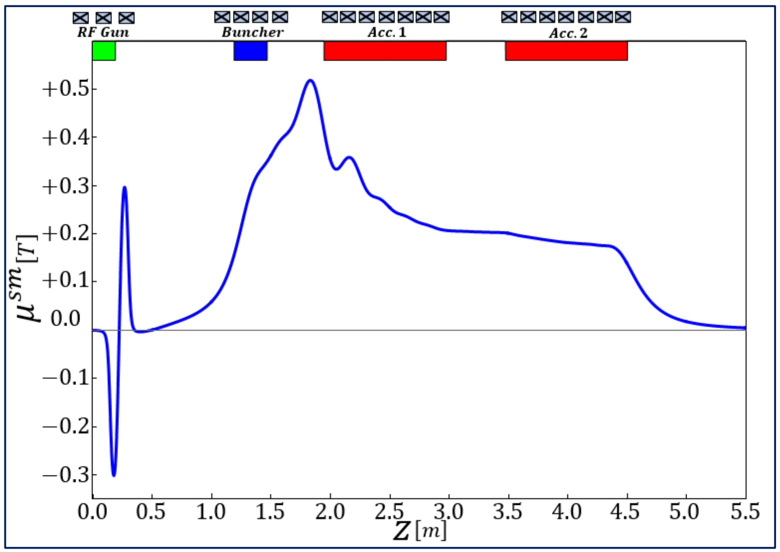
S Band RF Gun Very High Quality EBeam Generation

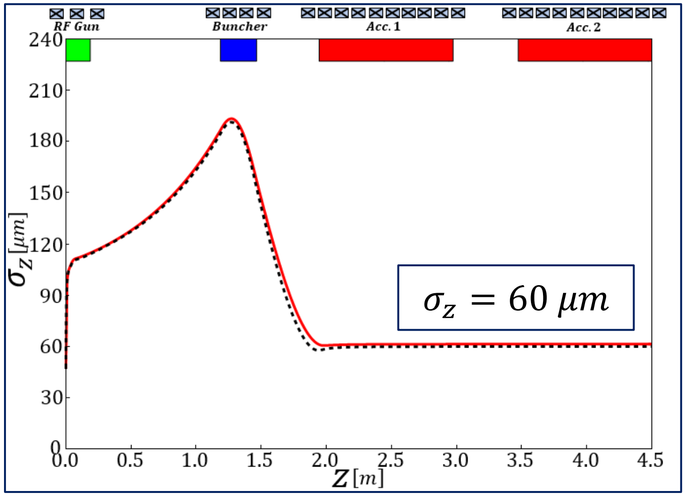
X Band Buncher for Very Strong Bunching

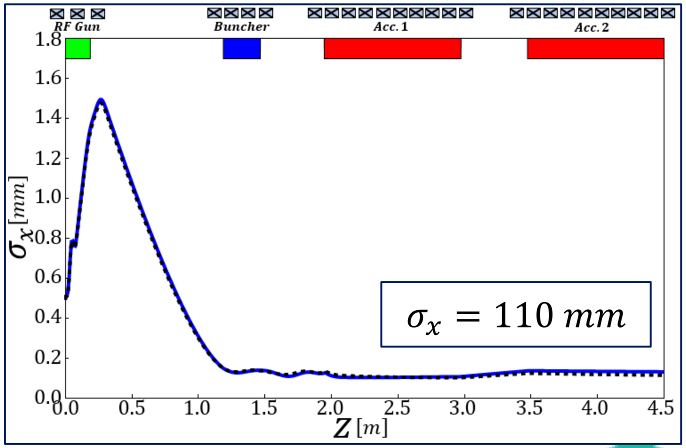
X Band Acc for Very Compact Acceleration

3.5 AWAKE Experiment

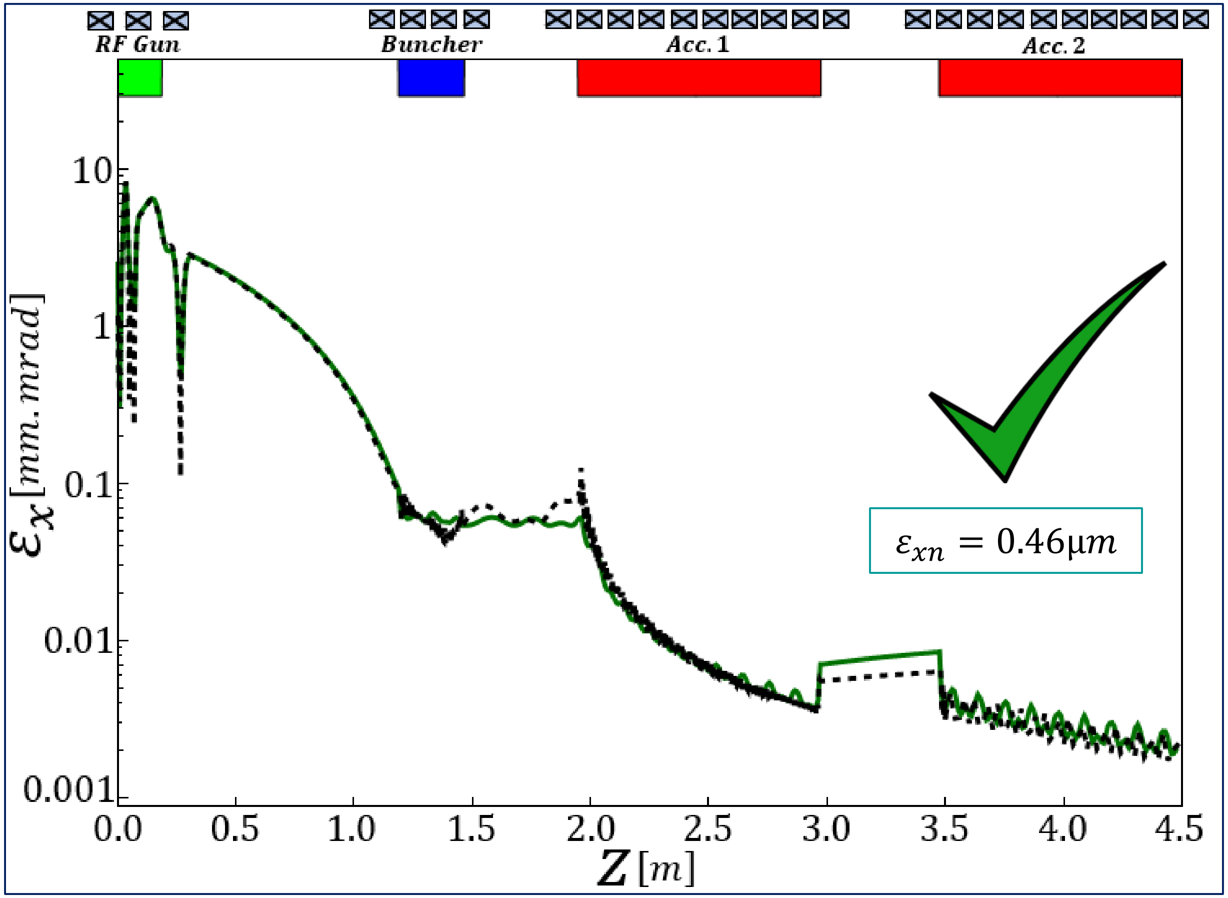
Parameter	$d_1[cm]$	$d_2[cm]$	$d_3[cm]$	$E_2[mV/m]$	$E_3[mV/m]$
Gradient	100	48	50	32	80

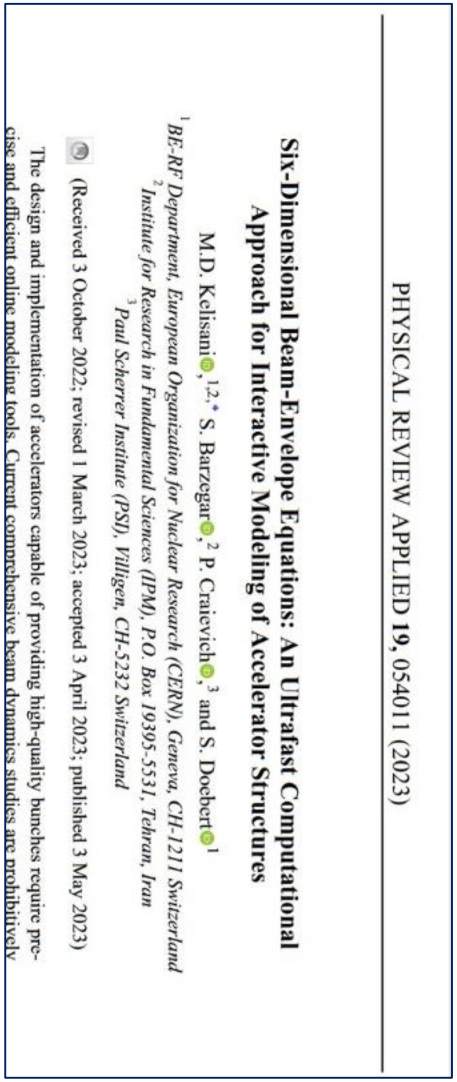






3.5 AWAKE Experiment




PHYSICAL REVIEW APPLIED **19**, 054011 (2023)

Six-Dimensional Beam-Envelope Equations: An Ultrafast Computational Approach for Interactive Modeling of Accelerator Structures

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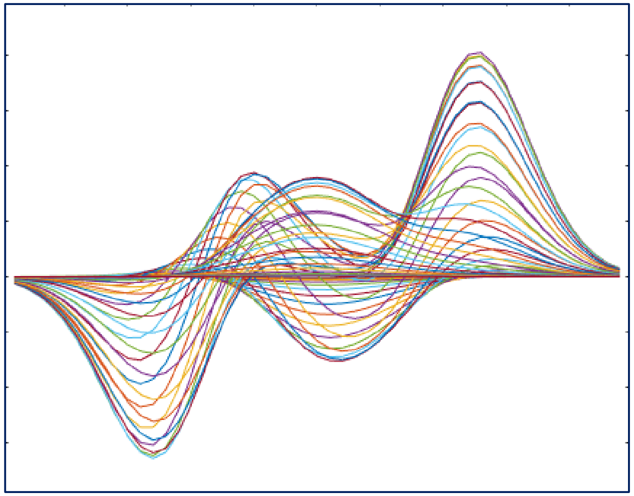
The design and implementation of accelerators capable of providing high-quality bunches require precise and efficient online modeling tools. Current comprehensive beam dynamics studies are prohibitively

Thanks for Attention

2.2 Perturbation



Perturbational Methods

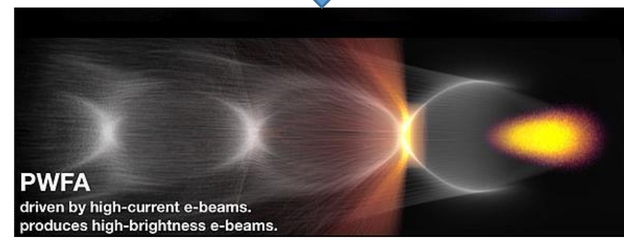
$$\beta_u = \beta_0(\hat{u}\,.\,\hat{z} + \Delta u')$$

$$\frac{1}{\gamma} \cong \frac{1}{\gamma_0}\left\{1 - {p_0}^2\Delta z' - \frac{{p_0}^2{\gamma_0}^2}{2}\Delta z'^2 - \frac{{p_0}^4{\gamma_0}^2}{2}\Delta z'^3 - \frac{{p_0}^2}{2}(1 + {p_0}^2\Delta z')\left(\Delta x'^2 + \Delta y'^2\right)\right\}$$

$$\frac{1 - \beta_z\beta_0}{\gamma} \cong \frac{1}{{\gamma_0}^3}\left(1 - 2{p_0}^2\Delta z' - \frac{{p_0}^2(1 - {p_0}^2)}{2}\Delta z'^2 - \frac{{p_0}^2}{2}\left(\Delta x'^2 + \Delta y'^2\right)\right)$$

$$\frac{\beta_0 - \beta_z}{\gamma} \cong \frac{\beta_0}{\gamma_0}\left(-\Delta z' + {p_0}^2\Delta z'^2 + \frac{{p_0}^2{\gamma_0}^2}{2}\Delta z'^3 + \frac{{p_0}^2}{2}\Delta z'\left(\Delta x'^2 + \Delta y'^2\right)\right)$$

3.5 AWAKE Experiment



Run 2c: Demonstrate Electron Acceleration and Emittance Preservation