New method for determining geometrical emittance using beam size and divergence measurements

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Outline

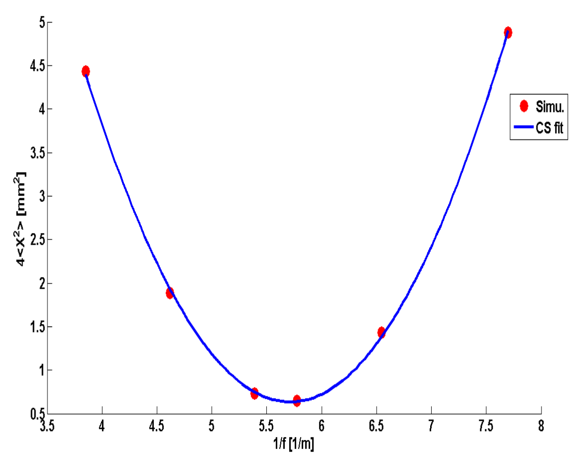
- Overview of emittance measurement techniques
- OTR measurement techniques to measure r and r'
- Using above to determine <rr'> and hence the geometrical emittance at beam size minimum applicable to emittance dominated beams
- Extending above method to determine <rr'> and geometric emittance for beams with significant space charge
- Simulation results
- Proposed experiments at PITZ to validate/apply new technique

Existing geometrical emittance measurement methods

$$\tilde{\epsilon}_r = \sqrt{\langle r^2 \rangle \langle r'^2 \rangle - \langle r r' \rangle^2}$$

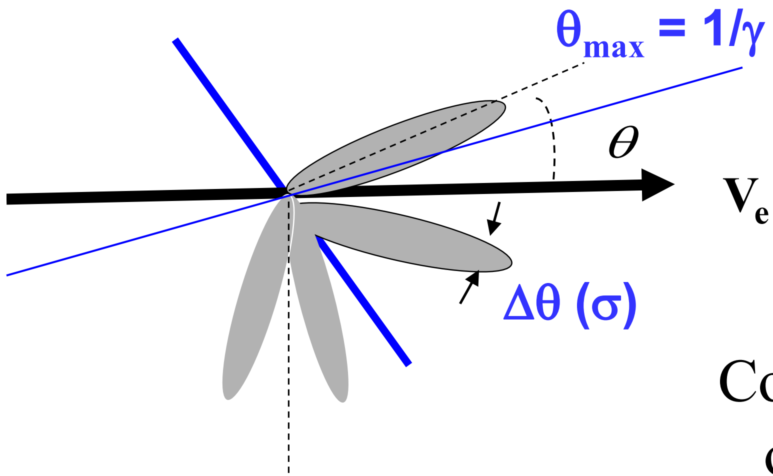
- **Pepper pot**: apertures used to sample smaller regions in transverse space
  works for beams with space charge
- **Quadrupole Scan**: most common method for measuring emittance; focal length of quadrupole is scanned and rms size beam data fit to quadratic (does not work well for beams with space charge)
- **Phase space tomography** (works for beams with linear space charge)
- **OTR technique** which measures r, r’ and uses these values to calculate the correlation term <rr’> :(works for emittance dominated beams and can be extended for beams with significant space charge)

Quad scan low space charge case



OTR angular distribution (AD) to measure beam Divergence*

NOTE: AD is independent of beam size or position



$$\frac{d^2I^{(S)}}{d\omega d\Omega} = \frac{e^2}{c\pi^2}\frac{\theta^2}{(\gamma^{-2}+\theta^2)^2},$$

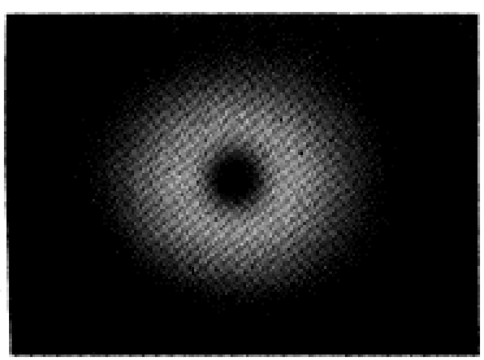
$$\theta \sim \quad \ll 1$$

Convolve AD with Gaussian distribution of trajectory angles: σ' = divergence

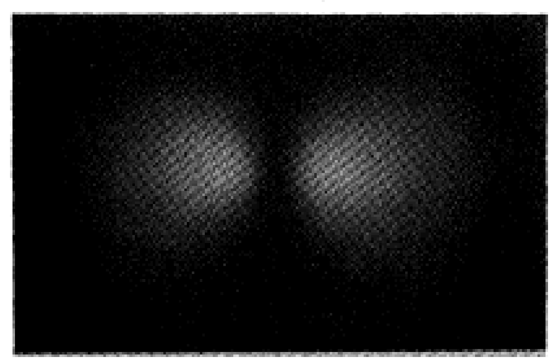
$\sigma' > 0.1/\gamma$

Radially Polarized AD Pattern Centered on Direction of V_e

Full AD



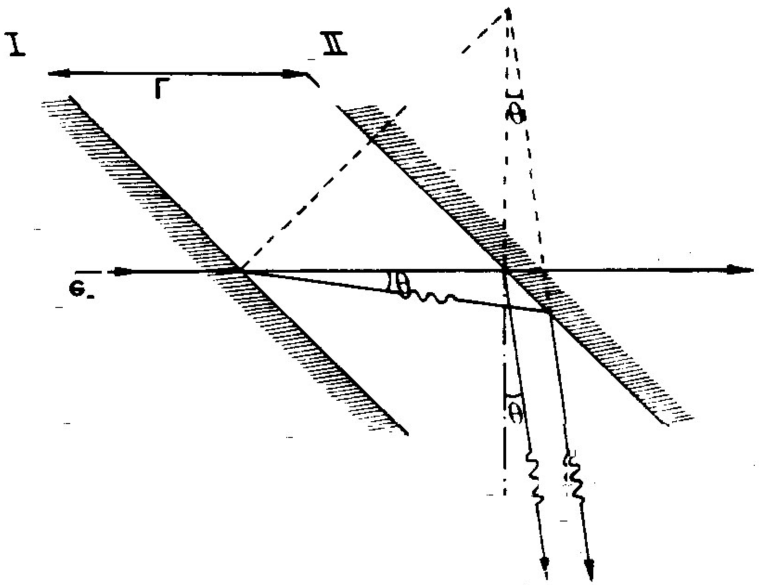
Horizontally Polarized AD



→ σ_x'

OTR Interferometry increases sensitivity to divergence

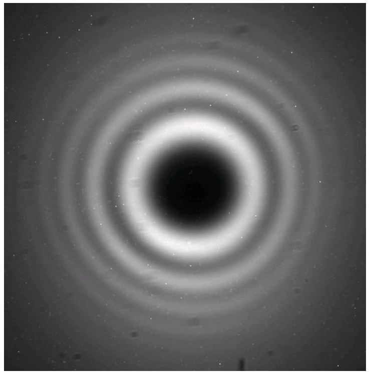
($\sigma' \sim 0.01/\gamma$, e.g 0.25 mrad at E=20 MeV)



$$\frac{d^2 I_{TOT}}{d\omega d\Omega} = \left[\frac{e^2}{c\pi^2} \frac{\theta^2}{(\gamma^{-2}+\theta^2)^2} \right] \left|1 - e^{-i\phi}\right|^2,$$

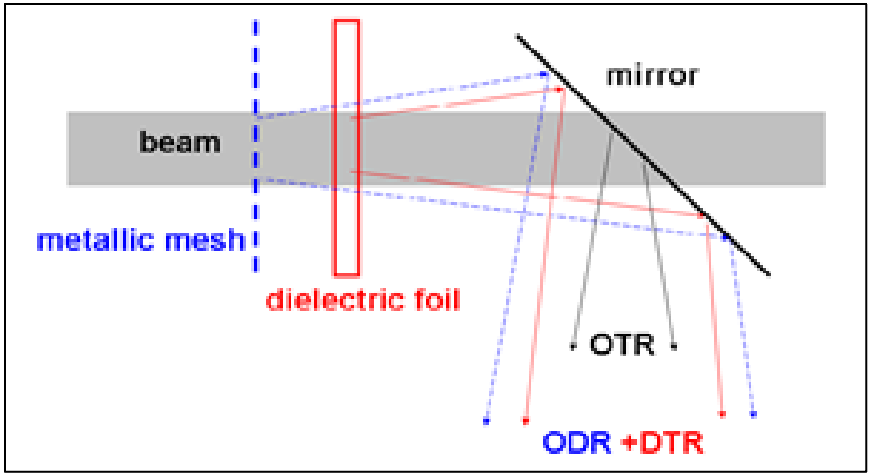
where: $\phi = L / L_V$,
(e-photon phase difference)

and: $L_V = (\lambda / \pi)(\gamma^{-2} + \theta^2)^{-1}$
(vacuum coherence length)

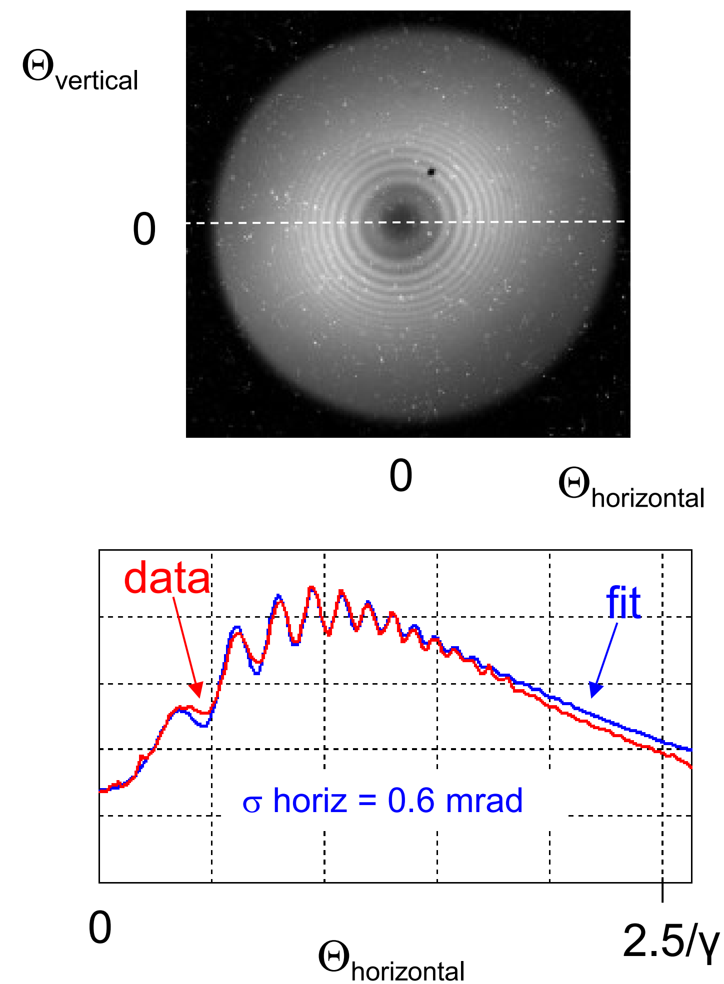


- Visibility of OTRI measures beam divergence *(and/or ΔE/E)*
- Radial Polarization of OTRI can be used to *separately* measure x' and y' *Fringe position measures beam energy (E)*

Low energy OTRI experiment done at Argonne/AWA*

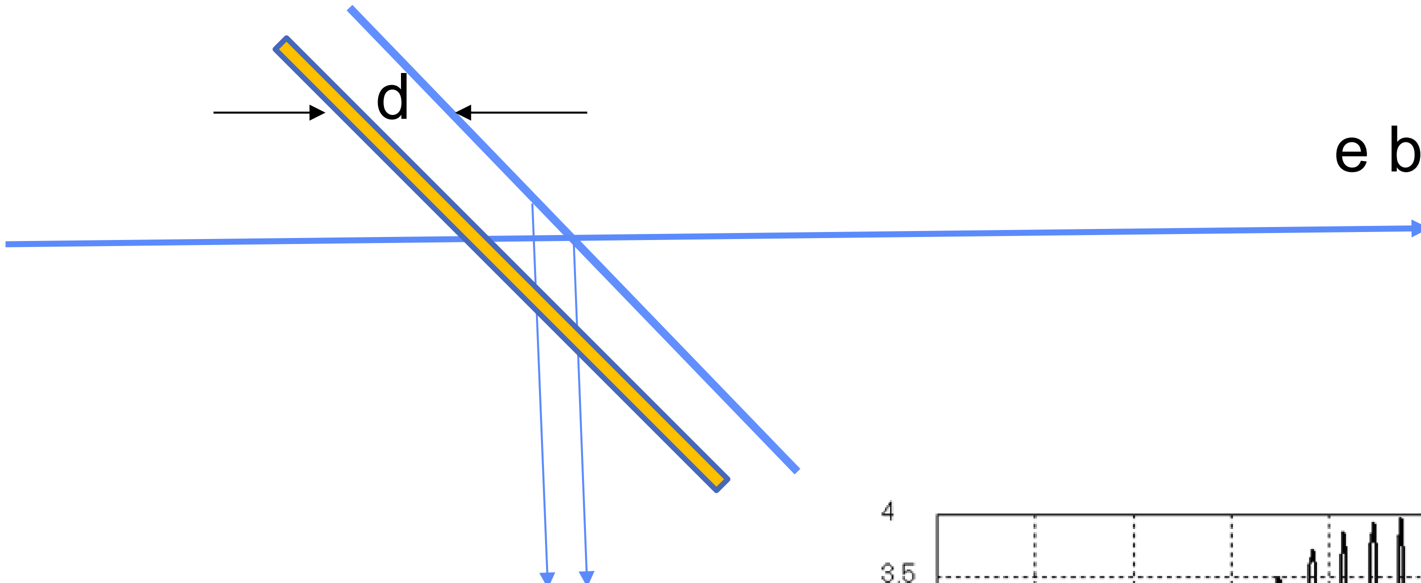


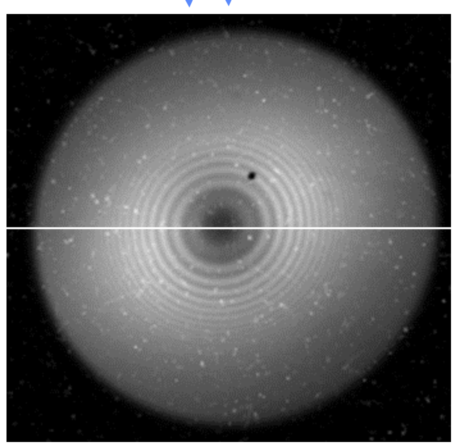
E =13.8 MeV, L =3.7 mm, λ = 632 x10nm,
Kapton foil: t = 9.15μ, index = 1.8

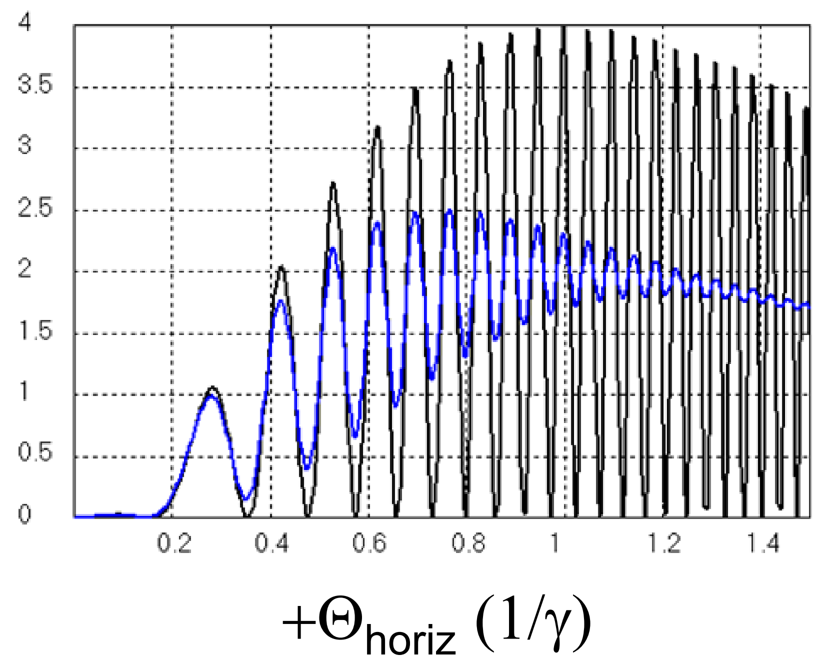


(Shkvarunets, et . al. BIW08)

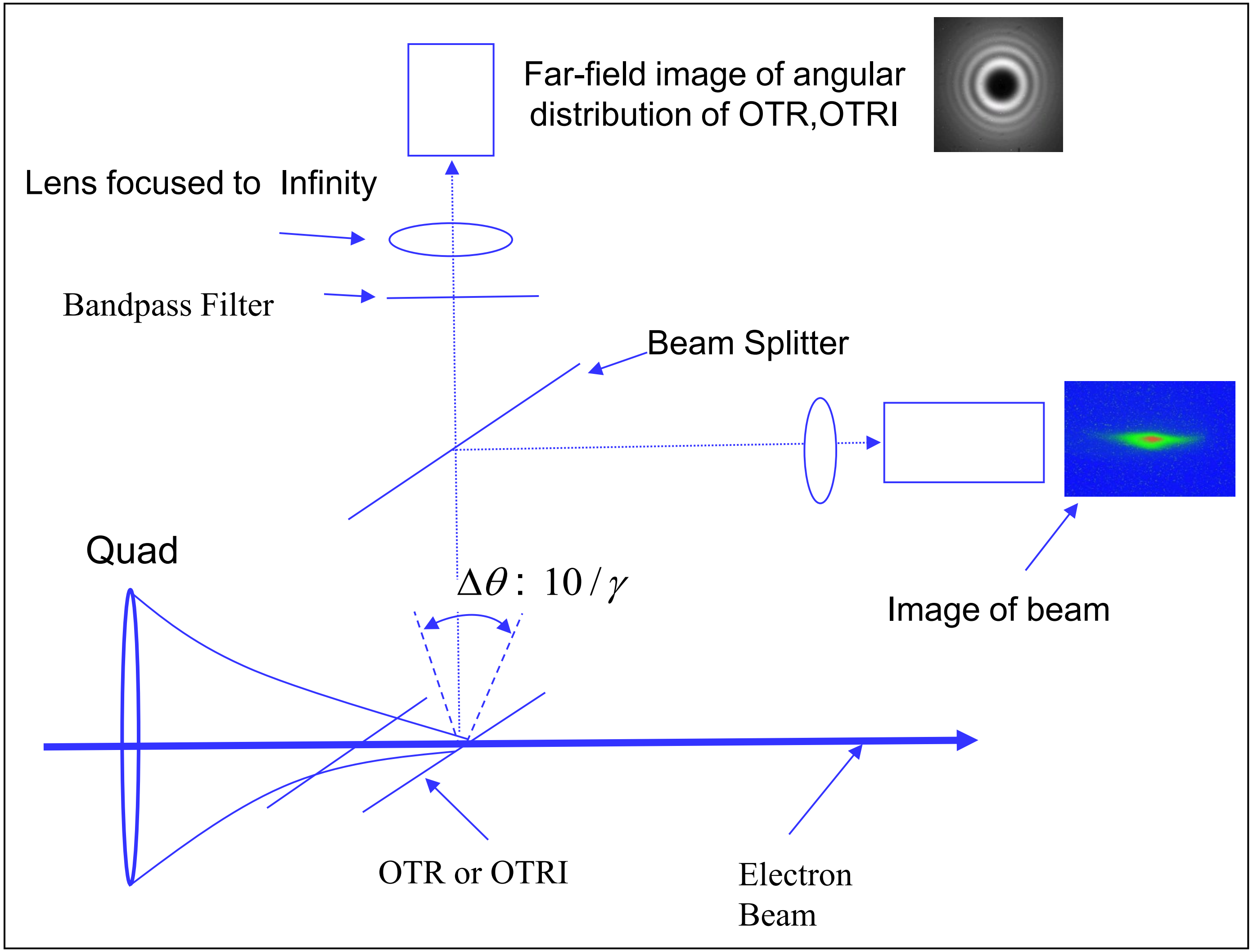
Dielectric foil (kapton)- mirror OTR interferometer optimized to measure $\sigma_x' \sim 0.4$ mrad; with d=20mm, λ = 635 nm







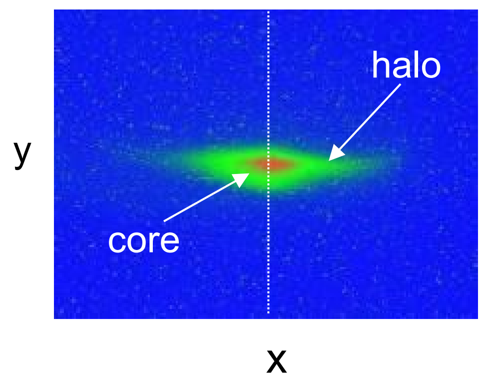
OTR method for simultaneously measuring σ and σ’



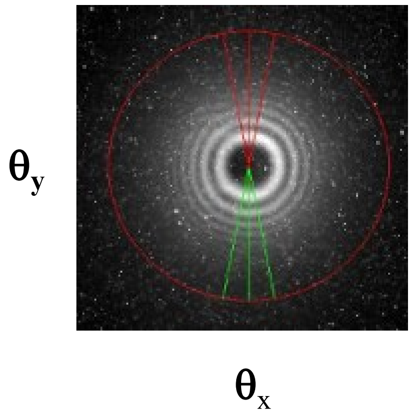
Example of beam size, divergences measurement at JLAB 135 MeV

(M. Holloway, et. al. PRSTAB '08)

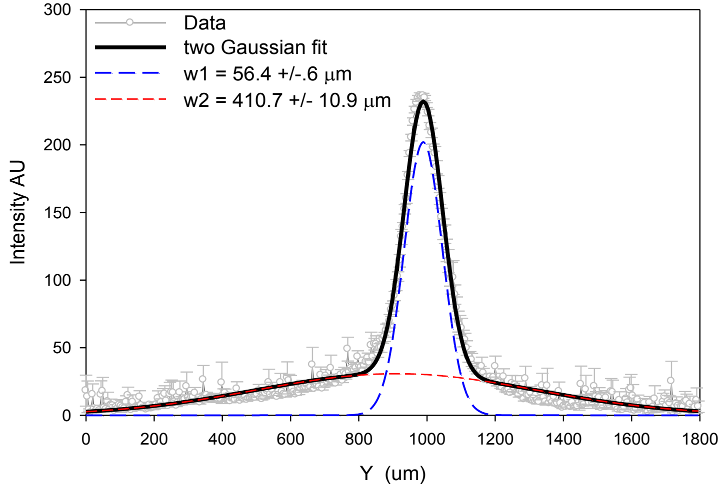
OTR Image of beam at y waist

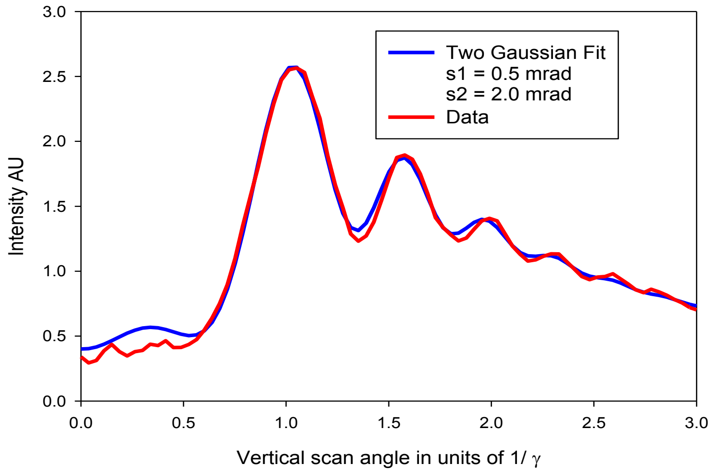


OTRI angular distribution



Vertical scan





Employing OTR measurements of σ and σ' to infer correlation term from these data to determine the geometric emittance

Calculation of correlation term for emittance dominated beam:

Apply transfer matrix description of quad scan

$$\begin{pmatrix} x_f \\ x'_f \end{pmatrix} = \begin{pmatrix} 1 & L \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -\frac{1}{f} & 1 \end{pmatrix} \begin{pmatrix} x_i \\ x'_i \end{pmatrix}$$

Final phase space coordinates | Drift section | Focusing Quad | Initial phase space coordinates

Taking moments, we get the equations for beam size, divergence and the cross-correlation term, as a function of f and L

[redacted]

Measuring cross-correlation term at beam size minimum

We find f_m, focal length for minimum beam size by solving:

$$\frac{d\left(< x_f^2 >\right)}{df} = 0$$

NB: $$\left.\frac{d\left(x(s,f)\right)}{ds}\right|_{f=f_0} = < xx' > = 0$$

Then *at this particular f* , the cross-correlation term is simply related to the minimum beam size via:

$$< x_f x'_f > = \frac{1}{L} < x_f^2 >$$

And therefore emittance can be derived as:

$$\epsilon_x^2 = < x_f^2 > \left(< x_f'^2 > - \quad \right)$$

Theory also predicts relative error in emittance measurements at beam size minimum neglecting <xx'>

Example: JLAB 100 MeV, OTR vertical emittance measurement*

$$\frac{\Delta\varepsilon}{\varepsilon} = \frac{1}{2}\frac{\langle yy'\rangle^2}{\langle y^2\rangle\langle y'^2\rangle} = \frac{1}{2}\frac{\langle y^2\rangle}{L^2\langle y'^2\rangle} \sim 4.4 \text{ x } 10(-4)$$

Caveats: 1) in general error may not be small

2) space charge forces neglected in this analysis

i.e. correlation term may be different - new approach to calculating <*rr'*> for beams with space charge required

New method to infer <rr'> that is applicable to emittance or space charge dominated beams*

Approach:

Use envelop equation in case of a quad or solenoid scan to compute cross-correlation term using beam size + divergence measurements at two focusing strengths ($1/f_{1,2}$).

*K. Poorrezaie, R. Fiorito, et. al. PRSTAB 2013

Advantages:

- Doesn't need a complete quadrupole or solenoid scan; only two pairs of r, r' data
- Multiple r, r' data pairs increase statistical accuracy of measurement

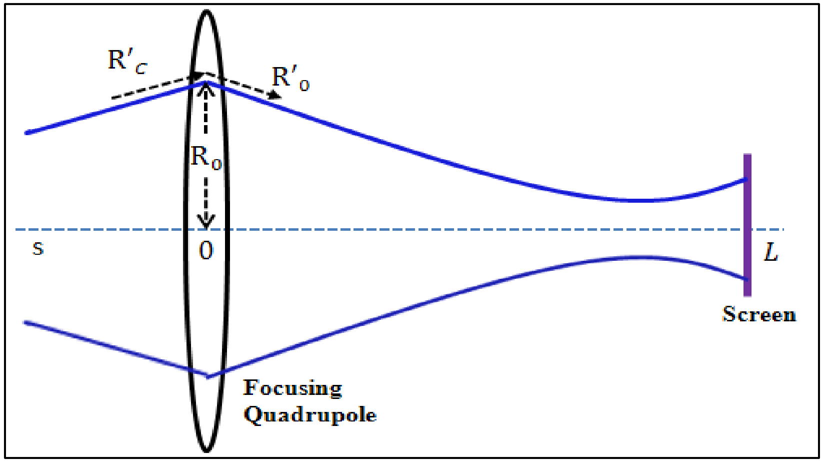
Limitations:

- Algorithm assumes linear space charge forces
- Round beam, symmetric focusing, e.g. solenoidal field
- Round beam, asymmetric focusing (e.g. quadrupole)
- Elliptical beam with any lens type

Approach: measuring emittance with two samples of radius/divergence

taken a two focal lengths f_1 and f_2:

$$\begin{cases} f_1 \\ <x_1^2> \\ <x_1'^2> \end{cases} , \quad \begin{cases} f_2 \\ <x_2^2> \\ <x_2'^2> \end{cases}$$



$$\begin{cases} f_1 \sim R'_{01} \\ f_2 \sim R'_{02} \end{cases}$$

Transfer matrix thin lens

$$\begin{bmatrix} R_0 \\ R'_0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -\frac{1}{f} & 1 \end{bmatrix} \times \begin{bmatrix} R_0 \\ R'_C \end{bmatrix} \Longrightarrow \boxed{\frac{1}{f} = \frac{R'_C - R'_0}{R_0}}$$

Example: round beam, symmetric focusing

- In linear Space Charge regime, beam envelope equation in drift region is:

$$\boldsymbol{R_j''(s) - \frac{K}{R_j(s)} - \frac{\epsilon_j^2}{R_j(s)^3} = 0 \qquad Eq.(1)}$$

where j is either of x or y. R denotes 2 x rms of x or y, and K is generalized perveance:

$$K = \frac{qI_b}{2\pi\epsilon_0 m(c\beta\gamma)^3}$$

- Beam parameters at lens can be calculated from beam parameters at foil by multiplying *Eq.(1)* by R', integrating Eq. (1) w.r.t. *s,* and solving the resultant Eq:

$$\boldsymbol{R'(s) = \mp\left[R(L)'^2 + \epsilon^2 \cdot \left(\frac{1}{R(L)^2} - \frac{1}{R(s)^2}\right) + 2K \cdot ln\left(\frac{R(s)}{R(L)}\right)\right]^{1/2} \qquad Eq.\ (2)}$$

No closed form solution to Eq. (2) -> numerical solution needed

Procedure for determining emittance

- **iterating until calculated emittance (in each step) converges toward beam emittance**

1. $XC1_1$, **initial guess for the cross-correlation term at f_1, is chosen.**

2. $XC2_n$ **, the cross-correlation term at f_2, is calculated as:**

$$XC2_n = \pm\sqrt{XC1_n{}^2 + A}$$

$\tilde{\epsilon}^2$ can be written in terms of beam parameters at each of the focal settings as:

$$\tilde{\epsilon}^2 = < x_i^2 >< x'^2_i > - XC_i{}^2 \quad (i = 1, 2)$$

This leads to:

$$XC_2{}^2 = XC_1{}^2 + A$$

Where A is:

$$A \equiv < x_2^2 >< x'^2_2 > - < x_1^2 >< x'^2_1 >$$

Procedure for determining emittance (cont.)

3. **Emittance and envelope slopes at L are calculated as:**

$$\begin{cases} \epsilon_n = 4[< x_1^2 > \cdot < x'^2_1 > - XC1_n{}^2]^{1/2} \\ R'_1(L)_n = 2\dfrac{XC1_n}{\sqrt{< x_1^2 >}} \\ R'_2(L)_n = 2\dfrac{XC2_n}{\sqrt{< x_2^2 >}} \end{cases}$$

$$R'(L) = 2\frac{d\sqrt{< r_L^2 >}}{ds} = 2\frac{< r_L \cdot r'_L >}{\sqrt{< r_L^2 >}} = 4\frac{< r_L \cdot r'_L >}{R(L)}$$

4. **Envelope equation can now be solved for finding envelope radius at lens and slope conditions for measurements 1 and 2:**

$$\begin{cases} R(0)_{1n} \quad , \quad R'(0)_{1n} \\ R(0)_{2n} \quad , \quad R'(0)_{2n} \end{cases}$$

$$\frac{1}{f_i} = \frac{R'_C - R'_{0i}}{R_0}$$

5. f_{1n} **estimate of focal length is given as:**

$$f_{1n} = \frac{f_2 \cdot R(0)_{1n}}{R(0)_{2n} + f_2 \cdot (R'(0)_{2n} - R'(0)_{1n})}$$

Updating cross-correlation at each step

6. f_{1n} is a function of $XC1_n$. i.e. $f_{1n} = g(XC1_n)$, and therefore $XC1_n$ is zero of this equation:

$$g(XC1) - f_1 = 0$$

a modified form of Newton method can be used to find $XC1_n$ as zero of this equation.

In first step ($n = 1$):

$$XC1_2 = 0.95\, XC1_1$$

Starting step 2 ($n \geq 2$), $XC1_n$ is updated according to:

$$XC1_{n+1} = XC1_n - \frac{XC1_n - XC1_{n-1}}{g(XC1_n) - g(XC1_{n-1})} \cdot (g(XC1_n) - f_1)$$

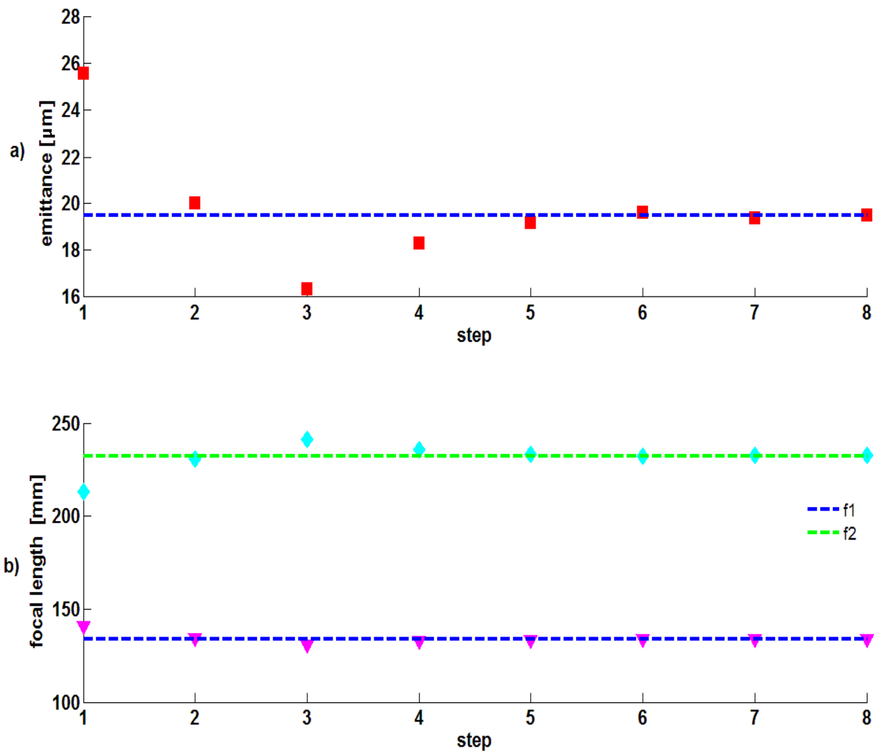
Numerical results for space charged beams

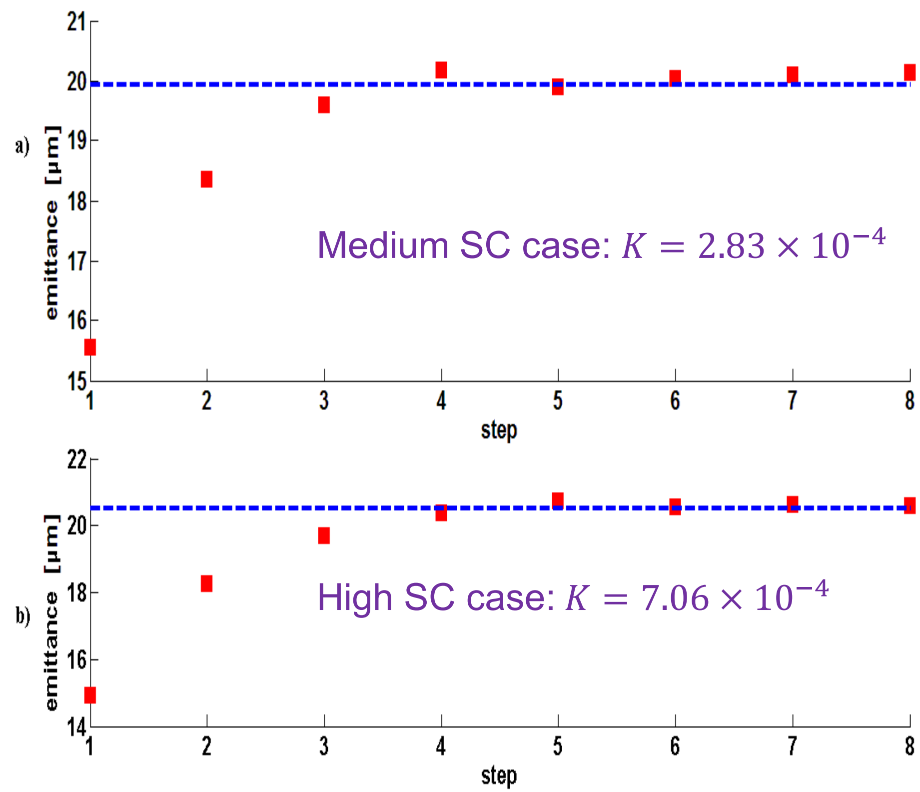
- **WARP was used to simulate the lens-drift experiment:**

$$\begin{cases} \epsilon = 19.5\ \mu m \\ R_0 = 3\ mm \\ L = 200\ mm \end{cases}$$

$$f_1 \begin{cases} \sqrt{< x_1^2 >} = 3.5\ mm \\ \sqrt{< x_1'^2 >} = 1.3\ mrad \end{cases} \qquad f_2 \begin{cases} \sqrt{< x_2^2 >} = 4.2\ mm \\ \sqrt{< x_2'^2 >} = 1.2\ mrad \end{cases}$$

Low SC case: $K = 8.82 \times 10^{-5}$



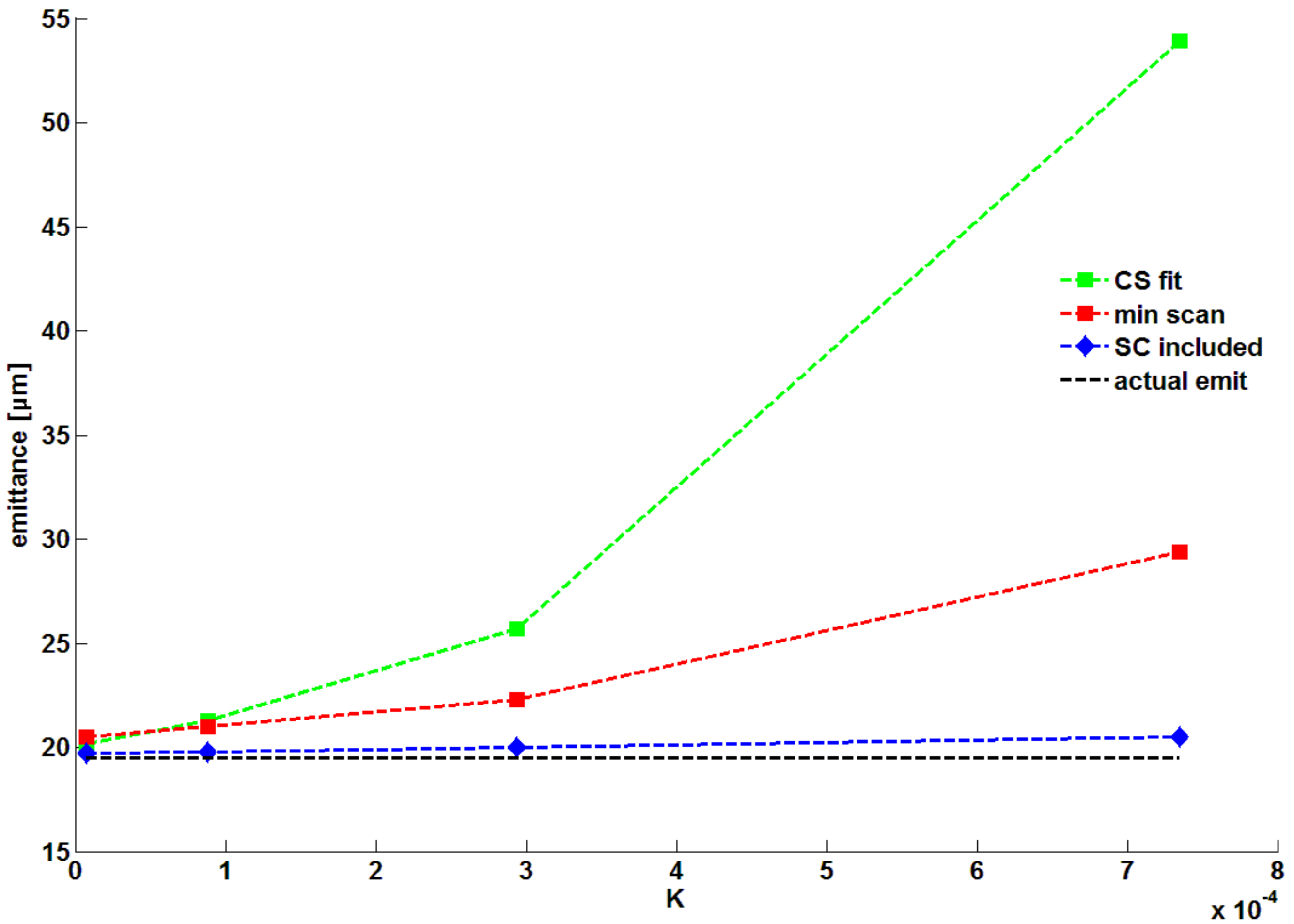


Medium SC case: $K = 2.83 \times 10^{-4}$

High SC case: $K = 7.06 \times 10^{-4}$

Comparison of various techniques to measure emittance for beams with space charge

- **Comparison of the emittance measured using different methods as a function of beam perveance (K) shows strength of the technique**



Proposed new emittance method measurements at PITZ

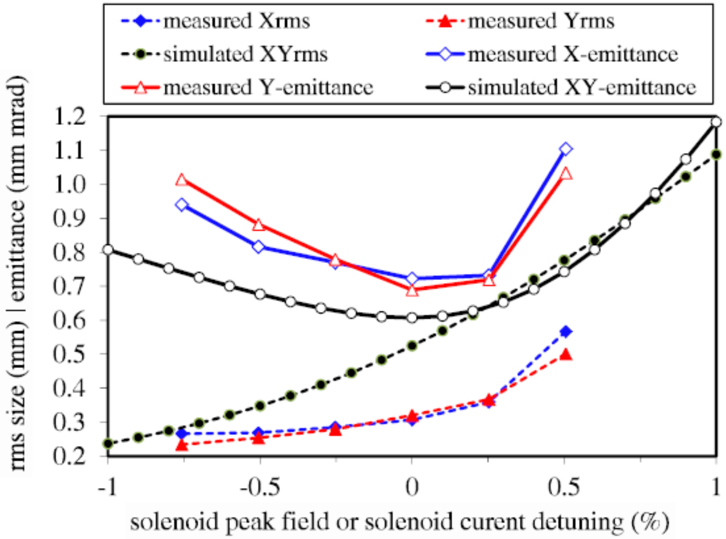
1) OTR measuremt of $\sigma_{x,y}$ *and* vs 1/f using solenoid or quadrupole scan

2) Use OTR data + new algorithm to compute $< xx' >$, $< yy' >$, ε_x , ε_y

3) Compare 2) with slit collimator and/or phase space tomography for previous measments and calculations for different [Q =0.1-2 nC]

4) Additional experiments that will take advantage of measuring σ'; and the novel algorithm and sparse data requirements to infer the cross correlation and emittance (your suggestions welcome).

e.g. Q=1nC, Fig. 20 PRSTAB 16, 2012)



single foil OTR regime

<xx'> simul.	emit. X meas.'d	sigma x meas'd.	sigma' x calculated	B
0.35	0.8	0.25	3.5	-0.5
0.42	0.75	0.25	3.4	-0.25
0.5	0.7	0.3	2.9	0
0.65	0.74	0.35	2.8	0.25
0.75	1.1	0.55	2.4	0.5

Errors:

Effect of measurement errors on emittance increase with space charge

1. Rms beam size: Dynamic range, resolution

2. Rms divergence: sensitivity of OTR AD pattern
   choice of technique, optimization of interferometer

   -> our analysis of tomography for linear space charge shows that err<13% and fits eqn.: $err = 3+9.8\chi^2$ where χ is estimated at lens

Other issues:

1. correlation term should be less than emittance
2. focal lengths of quad/solenoid should not be too close: minimum error achieved when: $1/f_1-1/f_2 > 0.1$
3. emittance should be constant in drift region: check with simulations: WARP, e.g.

.

END

Extra slides

Relating initial beam parameters to final parameters

How to write R_0 and R_0' in terms of R_L and R_L'?

- In envelope equation by replacing s with –s we get:

$$R(-s) = \left(R_0^{\ 2} - 2R_0 R'_0\, s + \left(\frac{\tilde{\sigma}}{R_0^{\ 2}} + \kappa_0 \right) s^2 \right)^{\frac{1}{2}}$$

- Plugging s=L we get:

$$R(-L) = \left(R_0^{\ 2} - 2R_0 R'_0\, L + \left(\frac{\tilde{\sigma}}{R_0^{\ 2}} + \kappa_0 \right) L^2 \right)^{\frac{1}{2}} \qquad R'(L) = 2\frac{d\sqrt{< r_L^2 >}}{ds} = 2\frac{< r_L \cdot r'_L >}{\sqrt{< r_L^2 >}} =$$

- It is equivalent to:

$$R_0 = R(0) = \left(R_L^{\ 2} - 2R_L R'_L\, L + \left(\frac{\tilde{\sigma}}{R_L^{\ 2}} + \kappa_L \right) L^2 \right)^{\frac{1}{2}}$$

- Likewise initial envelope slope is given as:

$$R'_0 = R'(0) = \frac{R_L R'_L - \left(\frac{\tilde{\sigma}}{R_L^{\ 2}} + \kappa'_L\right) L}{R_L}$$